

Hugh Blair Authr
THE
ELEMENTS of EUCLID:
As Rect WITH *104 Prop*
SELECT THEOREMS
OUT OF
ARCHIMEDES. 30/-

By the Learned ANDREW TACQUET.

To which are added.

PRACTICAL COROLLARIES, shewing the
Uses of many of the Propositions.

By WILLIAM WHISTON, M. A.

Mr. LUCAS's Professor of the MATHEMATICKS in the
UNIVERSITY of CAMBRIDGE.

In this TENTH EDITION is added an
APPENDIX of PRACTICAL GEOMETRY,
with Forty New Figures, and a Brief and Independent
Demonstration of certain Select and most useful Propositions.

By S. F.



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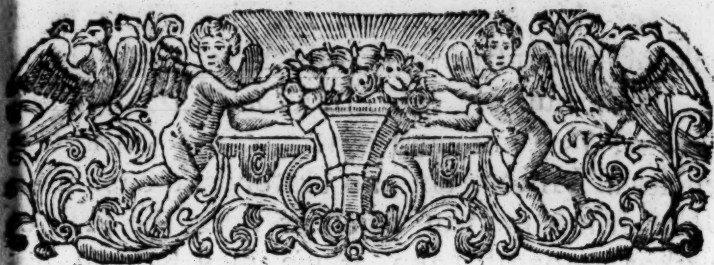
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A N
HISTORICAL ACCOUNT
OF THE
RISE and PROGRESS
OF THE
MATHEMATICKS.

IT seemed meet to me, when I was about to set forth the Elements of the *Mathematicks* to premise a few Things concerning the Rise and Excellency of this Science, that its Candidates may understand what a Kind of Science it is, to which they are about to dedicate themselves; and that it may be made manifest against those, who slight those Things, whereof they are ignorant of how great Value and Dignity this Knowledge is, which the wisest Men of all Ages have, with incredible Study, labour'd to attain unto, and become possess'd of. Moreover, I must own that *Peter Ramus's* Labours have been of great Service to me in the com-

piling of this Account, who in the whole First Book of his Institution, which is not a little one, hath out of *Proclus*, *Laertius*, *Gellius*, *Polybius*, *Izetzes*, and others, composed a Mathematical History both accurately and copiously.

The Mathematical Sciences were the first of all other amongst Men, if we may believe *Josephus*. He, *Book I. Chap. 3* writeth, That the Posterity of *Seth* observed the Order of the Heavens, and the Courses of the Stars. And lest these Inventions should slip out of the Knowledge of Men, *Adam* having predicted a two-fold Destruction of the Earth, one by a Deluge, the other by Fire, they raised two Columns, one of Bricks, of Stone the other; and inscribed their Inventions upon them, that if the Brick one should happen to be destroyed by the Deluge, that of Stone, which would remain, might afford Men an Opportunity of being instructed, and present to their View the Things which it had inscribed on it. They say also, that that Stone Pillar, which even in our Days is seen in *Syria*, was dedicated by them. This *Josephus* says; whom I leave to vouch for the Story.

That the *Assyrians* and *Chaldeans* were the first of Mortals, after the Flood, who applied themselves to the *Mathematicks*, is delivered by the same *Josephus*; as also by *Pliny*, *Diodorus* and *Cicero*. But the *Mathematick Arts* which first sprang among the *Chaldeans*, amongst whom they flourished, were afterwards transferred out of *Chaldea* and *Assyria* unto the *Egyptians*, by *Abraham*. For, when, at the Command of *G O D*, he went forth from his native Soil into *Palestine*,



Palestine, and from thence into *Egypt*, and perceiv'd the *Egyptians* to be taken with the Study of good Arts, and to be of a very notable Wit and Capacity for Learning, (as *Josephus* testifies, *Book I. Chap. 9.*) he communicated to them Arithmetick and Astronomy; and consequently Geometry, which must of Necessity go before Astronomy. In which Studies afterwards the *Egyptians* so flourished, that *Aristotle*, 1 *Metaph.* Chap. 1. doth affirm, that the *Mathematick Arts* were first found out in *Egypt*, by their Priests; who, by their Employments, were at leisure for these Things.

Then these Arts crossing the Sca out of *Egypt*, came to the Philosophers of *Greece*: For *Thales the Milesian*, who flourished 584 Years before Christ, was the first of the *Greeks*, who coming into *Egypt*, transferred Geometry from thence into *Greece*. He it was indeed, who, besides other Things, found out the 5th, 15th and 26th Propositions of the First Book. To the same are also owing the 2d, 3d, 4th, 5th, of the Fourth Book. The same Person began to observe the Equinoxes and Solstices, as *Laertius* testifies; and he was the first who foretold an Eclipse of the Sun, as *Hippias* and *Aristotle* do write; and *Tzetzes* saith, That he also foresaw'd an Eclipse of the Moon to King *Cyrus*. For which Thing's sake he is to be looked on as the first Founder and Author of the Mathematical Sciences in *Greece*.

After him was *Pythagoras* of *Samos*: Which most ancient Philosopher, exceedingly improved and adorned the Mathematick Sciences. And he so gave himself to Arithmetick in particular,

that almost his whole Method of Philosophizing was taken from Numbers. And he first of all, as *Laertius* relates, abstracted Geometry from Matter; in which elevation of the Mind, he found out the 32d, 44th, 47th and 48th Propositions of the first Book. But he is especially celebrated for the Invention of *Prop.* 32, and 47. of that Book; and he conceived so great Joy upon this Invention, that, as *Apollodorus* witnesses in *Laertius*, on that Account he sacrificed an Hecatomb. The same Person first laid open the Matter of incommensurable Magnitudes, and the Five regular Bodies. The same Person did both most diligently teach and exercise the Art of Astrology and Music: For he did not only acutely and subtilly find out many Things himself, but he also first opened a School, in which Youth might learn these honourable and noble Arts.

Pythagoras was followed by *Anaxagoras* of *Clazomene*, and *Oenopides* of *Chios*, of whom *Plato* makes mention in his Dialogue, *The Lovers*, where young Men are brought in contending about *Anaxagoras* and *Oenopides* in their Descriptions of Circles. *Aristotle* reports, that a certain Treatise of Geometry was written by *Anaxagoras*; and we have it from *Laertius*, that it was shewed by him that the Sun is greater than *Peloponnesus* (a notable Instance of the Infancy of Astronomy at that Time) and that he made some Conjectures concerning Habitations in the Moon. As for *Oenopides*, to him *Proclus* ascribes the 12th and 13th, *L. i.* These were followed by *Briso*, *Antipho*, and *Hippocrates* of *Chios*, all of them, for attempting the Quadrature

drature of the Circle, reprehended by *Aristotle*, and at the same Time celebrated. But amongst them, *Hippocrates* was by far the most Famous; that celebrated Person, who, of a Merchant, growing to be a Philosopher, and a Geometrician, besides the Quadrature of the Circle, also first attempted the Doubling of the Cube, by two mean Proportionals, which, as being an excellent, and indeed the only Way, all that have followed him to this Time have embraced it. 'Tis also his peculiar and great Commendation, that he, as *Proclus* testifies, first wrote Elements, and digested into Order the Discoveries made by others.

Democritus was admirable, not in Philosophy only, but also in the Mathematicks. His Physical Monuments, and, if such there were, his Mathematical Works also, are wholly lost, through the Envy (as some report) of *Aristotle*, who desired to have no other Writings read, but his own. The Philosophy of *Democritus* hath been restored by *Peter Gassendus*, in a most Learned Work lately put forth. *Theodorus Cyrenæus*, although none of his Mathematical Inventions are extant, yet is great upon this account, if there were no other, that he is reported to have been the Master of *Plato*.

Unto *Plato* therefore we are come at length, than whom no one brought greater Lustre to the Mathematical Sciences. He amplified Geometry with great and notable Additions, bestowing incredible Study upon it. And above all, the Art Analytic, or of Resolution, was found out by him, the most certain Way of Invention

vention and Reasoning. He set off and illustrated his Books of Philosophy in a Mathematical Way, and encouraged whatsoever was admirable in Mathematical Philosophy. Upon the Door of his Academy was read this Inscription: *οὐδεὶς ἀγνοήτωντος εἰσέρτω*: *Let no one ignorant of Geometry enter here*; an illustrious Instance to demonstrate, how the Mathematicks are not foreign, but proper, not unuseful, or unbecoming, but honourable and profitable to sound and certain Philosophy. In a Word, how great both Admirer and Master of the Mathematicks *Plato* was, that Man will of himself easily understand, who shall read his Monuments through.

Out of *Plato's* Academy, almost innumerable Mathematicians come forth. Thirteen of *Plato's* familiar Acquaintance are commemorated by *Proclus*, as Men by whose Studies the *Mathematicks* were improv'd. From hence were *Leodamus* the *Thasian*, *Archytas* the *Tarentine*, *Theætetus* the *Athenian*, by whom the *Mathematicks* were notably enlarged. *Leodamus* practised the Analysis received from *Plato*, and is said by *Lactertius* to have found out many things by the Help of it. As for *Theætetus*, both his own Inventions, amongst which are the Elements written by him, and the Inscription of regular Bodies; and *Plato's* Encomiums, who also inscribed a Dialogue to his Name, do make him famous.

Archytas also wrote Elements himself; and his Doubling of the Cube is mentioned by *Eutocius*; whose singular Commendation it likewise was, that he was almost the First that brought

brought down the *Mathematicks* to Human Uses; by whose Contrivance also a wooden Pidgeon was made to fly, as *Gellius* reports; he being preceded by *Dædalus*, and followed by other Artificers, yielded Matter for the Fables of the Poets. Moreover, *Archytas* was both a Mathematician and General of an Army: He five times commanded the Forces of his own Citizens, in the Wars of his Country, and five times overcame their Enemies. The meer Name of *Neoclides* is only famous, he being more illustrious for his Scholar *Leon* perhaps, than for his own Inventions. *Leon* certainly wrote Elements of all the *Mathematicks*, improved them, and made them more fit for Use. Wherefore he is deservedly to be reckoned amongst the chief Compilers of Elements.

Eudoxus of *Cnidos* was not inferior to *Leon*: A Man great in Arithmetick, and to him, (if we may believe the *Greek* Scholiast) we owe the whole Fifth Book. He likewise wrote Elements, and made them more general, and encreased the Sections begun by *Plato*; over and above this he was the first Framers of Astronomical Hypotheses, and derived down the Springs of Geometry, as *Archytas* had done before, to Mechanics. *Amyclas* the *Heracleot*, and *Menæchmus*, and his Brother *Dinostratus*, *Helicon* of *Cyzium*, *Theudius*, *Hermotimus* the *Colophonian*, *Philippus* the *Medmean*, all Platonists rendered Geometry much more perfect. *Menæchmus* also found out the Conic Sections, and by the help of them, two mean Proportionals; whose Invention in this Case is preferred by *Eutocius* before
any

any other. *Theudius* and *Hermotimus* made the Elements more universal and full. And all these, who were of *Plato's* Academy, brought Mathematick Philosophy to Perfection, as *Proclus* saith. *Xenocrates* also, one of *Plato's* Auditors, and Master of *Aristotle*, as well as *Aristotle* himself, were famous for the Knowledge of the Mathematicks. When a certain Person, who knew nothing of Geometry, was minded to be his Auditor, *Go thy Way*, saith he, *for thou wantest the very Handles of Philosophy.*

But of *Aristotle*, what can I say; All his Books are filled with Mathematical Arguments, out of a Collection of which *Blancane* hath made a Book. Two of *Aristotle's* School are especially celebrated, *Eudemus* and *Theophrastus*: This latter wrote two Books of Numbers, four of Geometry, and one of indivisible Lines: The other, composed a Mathematical History; and from him *Proclus*, and others have borrowed theirs. To *Aristeus*, *Isidore*, *Hipsicles*, most subtil Geometricians, we are especially indebted for the Books of Solids. Lastly, *Euclid* gathered together the Inventions of others, disposed them into Order, improved them, and demonstrated them more accurately, and left to us those *Elements*, by which Youth is every where instructed in the Mathematicks. He died in the Year before Christ, 284. There followed *Euclid* almost an 100 Years afterwards *Eratosthenes* and *Archimedes*. The Name of *Eratosthenes* was very famous, but his Writings are lost. Many Remains we have of *Archimedes*, and many we have lost.

But

But when I name *Archimedes*, I conceive in my Mind the very Top of Human Subtilty, and the Perfection of the whole Mathematical Sciences. His wonderful Inventions have been delivered to us by *Polybius*, *Plutarch*, *Tzetzes* and others. *Conon* was Contemporary to *Archimedes*, one who was both a Geometrician and an Astronomer, whose Death *Archimedes* laments in his Book of the Quadrature of the *Parabola*. There followed *Archimedes* and *Conon*, and that at no great Distance, *Apollonius* of *Perga*, another Prince in Geometry, who was called by way of high Encomium, *The Great Geometrician*. There are extant Four [now Seven] most subtil Books of his *Conic's*. To the same Person are ascribed the Fourteenth and Fifteenth Books of *Euclid*, which were contracted by *Hypficles*. *Hipparchus* and *Menelaus* wrote, this latter, Six, the other, Twelve Books of Subtenses in a Circle; for which Invention so very profitable and necessary, great Commendations and Thanks are due to both. There are also extant three Books of *Menelaus* concerning Spherical Triangles. Three most useful Books of Spherics of *Theodosius* the *Tripolite* are also in the Hands of all. And these indeed, if you except *Menelaus*, lived all of them before Christ.

In the Year after Christ, 70, there appeared in the World, *Claudius Ptolomæus*, the Prince of Astronomers, a Man certain'y wonderful, and (as *Pliny* saith) above the Nature of Mortals. But he was not only most skilful in Astronomy, but in Geometry also; which, as many other Things

Things written by him do witness, so especially do the Books of Subtenses: Those of *Menelaus*, which were Six, and the Twelve of *Hipparchus*, all contracted by him into five Theorems. As for *Plutarch*, a most famed Philosopher, there are extant his Mathematical Problems. And all know of the learned Commentaries of *Eutocius* the *Ascalonite* upon *Archimedes*. By him are recited the Inventions of *Philo*, *Diocles*, *Nicomedes*, *Sphorus*, *Heron*, as of so many excellent Masters in the Mathematicks, concerning Doubling the *Cube*. *Heron's* Genius certainly was excellent, as well for Mechanicks as Geometry. The Doubling the *Cube* delivered by him, is commended by *Pappus*, Book III. *Prop.* 7. before all other. The admirable Works of *Ctesibius* the *Alexandrian*, to whom we owe our Pumps, are celebrated by *Vitruvius*, *Proclus*, *Pliny* and *Athenæus*. The Name also of *Geminus* is not in the lowest Place amongst Mathematicians, whom *Proclus* has preferred in many Things before *Euclid* himself.

Diophantus, and he also an *Alexandrian*, was as great in Arithmetick, as *Archimedes*, *Apolleinius* or *Euclid* in Geometry; he was certainly a Master of all Subtilty relating to Numbers: By him was found out that admirable Art, which we call *Algebra*; which in these Times has been rendered more perfect and universal by *Francis Vieta*, and *Renatus Cartesius*. There are others who are celebrated amongst the Antients also; as *Nicomachus*, famous for Arithmetical, Geometrical and Musical Monuments; *Serenus* well known to Geometricians for his two Books, concerning

concerning the Section of a Cylinder; *Proclus*, *Pappus*, *Theon*. How great a Mathematician *Proclus* was, is manifest from his learned Commentaries on *Euclid*, and other Writings. And this is he, I suppose, who, as *Zonarus* reports, and from him *Ramus* and *Baronius*, about the Year of Christ 514, with Optic Artifice, and the Glasses which he used, burnt the Fleet of *Vitalian*, who was besieging *Constantinople*. The Praises of *Theon*, which truly are deservedly great, *Peter Ramus* wonderfully exaggerates; insomuch, that even the Books which hitherto all have ascribed to *Euclid*, ought, as he thinks, to be attributed to *Theon*. But *Ramus*, who every where is ready to detract from *Euclid*, and this without grounding himself upon any solid Foundation, is not to be hearkened to here. To come at length to a Conclusion, let *Pappus* bring up the Rear, the last in Time amongst the Antients, as being one who lived about the Year 400; but in Reputation, and all Mathematical Commendation, to be reckoned amongst the first. *Alexandria*, that City so fruitful of great Men, which before had brought forth *Hypsicles*, *Ctesibius* and *Diophantus*, produced him also, to the great Advantage of the Mathematicks. He wrote Seven Books of Mathematical Collections, of which the Two first are lost. The Five other do abound with so many, and such various most noble Inventions in almost all Parts of the Mathematicks, that they are esteemed amongst the chief Monuments of the Antients which are extant.

And

And thus you have a short History of the Origin and Progress of the Mathematicks. From which appears the Antiquity, Excellency and Dignity of this Science. Certainly the same eminent Persons in the Commonwealth of Learning, who discovered Philosophy, discovered also the Mathematicks, like two Sisters born at one Birth; whom, if any one would violently separate from each other, he certainly attempts to break off their Native Concord, with most notable Injury, and as it were Cruelty to both; seeing, as it is wont to fall out in the Case of Twins, where they are removed from one another, in Place or by Death, so it will be like to happen here, that Mathematicks being plucked away from her, Philosophy must needs languish and pine away.

N. B. *Q. E. D.* or *W. W. D.* is, which was to be demonstrated.

Q. E. F. which was to be done.

Q. E. I. which was to be found.





Doctor *Barrow's* Words prefix'd
before his *Apollonius*.

G O D always acts Geometrically.

H O W great a Geometrician art thou, O Lord! For, while this Science has no Bound; while there is for ever Room for the Discovery of new Theorems, even by Human Faculties, Thou art acquainted with them all at one View, without any Train of Consequences, without any wearisome Application of Demonstration: In other Arts and Sciences our Understanding is able to do almost nothing; and like the Imagination of Brutes, seems only to dream of some uncertain Propositions; Whence it is, that in so many Men are almost so many Minds: But in these Geometrical Theorems all Men are agreed: In these the Human Faculties appear to have some real Abilities, and those Great, Wonderful and Amazing. For those Faculties which seem of almost no Force in other Matters, in this Science appear to be Efficacious, Powerful and Successful, &c. Thee therefore do I take hence Occasion to Love, and Rejoice in, and Admire; and to pant after that Day, with the Earnest Breathings of my Soul, when thou shalt be pleased,

B

out

Dr. BARROW.

out of thy Bounty, out of thy Immense and Sacred Benignity, to grant me the Favour to perceive, and that with a pure Mind, and clear Vision, not only these Truths, but those also, which are more numerous, and more important; and all this without that continual and painful Application of the Imagination, which we discover these withal, &c.

Mathematical Notes, or Abbreviations.

= The Note for Equality. So $a = b$ signifies that a and b are equal.

+ The Note for Addition. So $a + b$ signifies the Sum of a and b together.

— The Note for Subtraction. So $a - b$ signifies the Difference between a and b .

$\times$ The Note for Multiplication. So $a \times b$, or $a b$ signifies a multiplied by b .

: : The Note for Equality of Proportion. So $A : B :: a : b$ signifies that A bears the same Proportion to B , that a bears to b .

$\div$ The Note of continued Proportion. So $A B C \div$ signifies that A bears the same Proportion to B , that B bears to C .

q The Note for a Square. So $C B q$ signifies the Square of the Line $C B$.

c The Note for a Cube. So $C B c$ signifies the Cube of the Line $C B$.





T H E

Elements of EUCLID.

BOOK I.

DEFINITIONS.

A Point is a Mark in Magnitude, which is [supposed to be] indivisible.

That is, which cannot be divided so much as in Thought. A Point is the Beginning, as it were, of all Magnitude, as Unity is of Number.

2. A Line is a Magnitude which hath Length only, and wants all Breadth; forasmuch as it is understood to be produced from the flowing of a Point.

3. Points are the Terms of a Line.

4. A right Line is that which lies evenly betwixt its Terms.

Or, as *Archimedes*; a right Line is the least of all those which have the same Terms; or, is the shortest of all those which can be drawn betwixt two Points.

Or, as *Plato* hath it; a right Line is that whose Extremes hide all the rest: [That is, when the Eye is placed in a Continuation of the Line.]

The Sense is the same in all. The Instrument whereby right Lines are described, is [called] a Rule; which, whether it be strait or not, you may know by this Tryal.

Describe a Line, according to the Rule; then, turning the Rule so that, that which was before the Right-hand End may now become the Left-hand End, apply it again

Fig. 1.
Table 1.

to the Line before described; if it doth now entirely fall in with the Line, the Rule is strait; if not, the Rule is not strait. The Reason hereof depends on *Axiom* 13.

5. A Surface is a Magnitude which hath only Length and Breadth.

It hath two Dimensions therefore: And is understood to be produced by the flowing of a Line.

6. Lines are the Extremes of a Surface.

7. A Plane, or a plain Surface, is that which lies evenly betwixt its extreme Lines.

Or, as *Hero*, that, to all the Parts whereof a right Line may be accommodated.

For it is produced from the Motion of a right Line.

Or, a plain Surface is that whose Extremes any of them hide all the Rest, [the Eye being placed in a Continuation of the Surface.]

Or, it is the least of all Surfaces, which have the same Terms. The Sense is the same in all.

Euclid hath not here defined a Body or Solid, because he was not yet about to treat concerning it. But lest any one should want the Definition thereof, take it here thus: A Body is a Magnitude, long, broad and deep. A Body therefore, hath three Dimensions; a Surface two; a Line one; a Point none.

8. A plain Angle is the mutual Inclination to each other of two Lines, which touch one another in a Plain, and so as not to make one Line.

Fig. 2. 4. Therefore the two Lines A B, C A touching one another in A, but so as not to make one Line, constitute an Angle.

9. The Sides, or Legs of an Angle are the Lines which make the Angle.

10. The *Vertex* or Top of an Angle is the Point (A) in which the Legs do meet, and touch one another.

Note, That a single Angle is designed by one Letter put at the Top: When there are more at one Point, they are design'd by three Letters, the middlemost of which denotes the Top of the Angle; and many Times also by one Letter interpos'd betwixt the Sides near the Top. So in *Fig. 5.* the Angle made by the Lines B A, C A is designed either by three Letters B A C, or by one only O.

11. Angles are called equal, if when the Tops of them are laid upon one another, the Sides of one agree with the Sides of the other. But unto this it is not required that the Sides should be of an equal Length.

12. They



12. They are called Unequal, when, the Top and one Side agreeing, the other doth not agree and that is called the Greater, whose Side falls without. So the Angle B A E is greater than the Angle B A C.

Fig. 5.

An Angle is not diminish'd or increas'd by the Diminution or Augmentation of the Sides that include it.

13. A right lin'd Angle is that which right Lines constitute; a curvi-lineal, which crooked Lines; a mixt one, that which a right Line and a crooked one make.

Fig. 2. 4.

14. When the right Line [C A] standing upon the right one [B F] leans unto neither Part, and therefore makes the Angles on both Sides equal, $C A B = C A F$ both of the equal Angles are called Right ones: But the right Line C A which stands upon the other, is called a perpendicular Line, or barely a Perpendicular.

Fig. 6.

A right Angle may also be defined thus.

A right Angle is that, that (B A C) when on the other Side an equal one ariseth (C A F) if you produce or draw forth a Side, as (B A)

Two Rules so joined as to contain a right Angle, make an Instrument, which is called a Square. *Pythagoras* was the Inventor of it, as *Vitruvius* affirmeth c. 2 l. 9. So great is the Use and Force of a right Angle in Framing, Measuring and Strengthening all Things, that nothing almost can be done without it. The Proof of a Square is made thus: Apply the Side of it, A E, to the right Line A F, and describe the right Line C A along the other Side, Then turning the Square towards B, if on both Sides it agrees to the right Line C A, A B, you may know that it is true and exact. The Reason hereof appears from the 14th *Definition* itself.

15. The Angle B A C, which is greater than the right one F A C, is called an obtuse Angle.

Fig. 7.

16. The Angle (L A C) which is less than the right Angle (F A C) is called an Acute one.

Fig. 8.

17. A plain Figure is a plain Surface, bounded on every Side with one or more Lines.

18. A Circle is a plain Surface, contained within the Compass of one Line called the Circumference; from which Line all the right Lines that can be drawn unto one certain Point, within the contained Space [A] are equal.

Fig. 9.

19. That Point is called the Center.

20. The Diameter is a right Line [B A] drawn thro' the Center, and on both sides ended at the Circumference;

Fig. 9.

and consequently it divides the Circle into two equal Parts. (As is abundantly manifest from the exact Agreement of two Semi-circles when laid upon one another.)

21. The Semi-diameter, or *Radius*, is the right Line A F, drawn from the Center to the Circumference.

22. The Semi-circle is a Figure (B L C) which is contained by the Diameter B C, and half the circumference (B L C)

Mathematicians are wont to divide the Circumference into 360 equal Parts (which they call Degrees) the Semi-circumference into 180, the Quadrant, or Quarter, into 90.

23. A Right-lin'd Figure, is a plain Surface bounded on every side with right Lines.

Fig. 10. 24. A Triangle is a plain Surface contained by Three right Lines.

This is the first and most simple of all Right-lin'd Figures and that into which they are all resolved.

Fig. 10. 25. An Equilateral Triangle, is that which hath all the Sides equal.

Fig. 11, 12. 26. An *Isoceles*, or Equicrural Triangle, is that which hath only two Sides equal.

Fig. 13. 27. A *Scalenum*, is that which hath Three unequal Sides.

Fig. 13. 28. A Right-angled Triangle, is that which hath one Angle right.

Fig. 12. 29. An Obtuse-angled Triangle, is that which hath One obtuse Angle.

Fig. 10, 11. 30. An Acute angled Triangle, is that which hath Three acute Angles.

Fig. 14, 15. 31. Amongst Quadrilateral Figures, the Rectangle is that which hath four right, and consequently equal Angles; whether the sides be equal or not.

Fig. 15. 32. A Square, is that which hath equal sides, and is Right-angled, and consequently Equi-angled.

Every Square is a Rectangle; but every Rectangle is not a Square.

Fig. 16. 33. A Rhombus is a Quadrilateral, or four-sided Figure, which is Equilateral, but not Equi-angled.

Fig. 17. 34. A Rhomboides, is that which hath the opposite sides and Angles equal, but is neither Equilateral, nor Equi-angled.

Fig. 14, 15, 16, 17. 35. A Parallelogram, is a Quadrilateral Figure, which hath each Two of its opposite sides (A B, F C and B F, A C) parallel

parallel to each other. Now what parallel Lines are, will be shewed in the following *Definition*.

Every Rectangle and Square is a Parallelogram; but every Parallelogram is not a Rectangle or a Square.

36. Right Lines are parallel, or equi-distant, which being *Fig. 18.* in the same plane, and drawn out on both sides infinitely, are distant from one another by equal Intervals.

The Intervals are said to be equal; in respect of the Perpendiculars. Wherefore if all the perpendiculars (Q L) unto one of the two Parallels (A B) shall be equal, the right Lines (A B, C F) are said to be parallel.

Parallels are produced, if the right Line (L Q) which is perpendicular to the right Line (A B) be moved along (A B) always perpendicularly; for then its Extremities L describes the Parallel C F.

37. The Diameter, or Diagonal of a Parallelogram, *Fig. 17.* and every Quadrilateral, is a right Line (A F) drawn through the opposite Angles.

38. Plain Figures contained by more sides than Four, are called Many-sided, or Many-angled, and by a *Greek Word, Polygons.*

39. The external Angle of a Right-lin'd Figure, is that *Fig. 19.* which ariseth without the Figure, when the Side is produced. Such are F B C, G C A, H A B. Every Figure therefore hath so many external Angles as it hath Sides, and internal Angles.

Postulates.

A Postulate is that which is manifest in itself, that it may easily be done, or conceived to be done. It is required therefore to be granted that we may,

1. From any Point given draw a right Line unto any other Point given.
2. Draw forth a finite right Line in Length still farther.
3. From any Center at any Interval describe a Circle.

Axioms.

Axioms.

AN *Axiom* is a Truth manifest of itself.

1. Those Things which are equal to the same thing, are equal also amongst themselves. And that which is greater or less than one of the Equals, is also greater or less than the other of them.

2. If to Equals you add Equals, the Wholes will be equal.

3. If from Equals you take away Equals, the Remainders will be equal.

4. If to Unequals you add Equals, the Wholes will be unequal.

5. If from Unequals you take away Equals, the Remainders will be unequal.

6. What Things are each of them half of the same Quantity, are equal amongst themselves; and what things are double or treble, or quadruple of the same, are equal amongst themselves.

7. What things do mutually agree with one another are equal.

8. If right Lines be equal, they will mutually agree with one another; and the same thing is true of Angles.

9. The Whole is greater than its Part.

10. All right Angles are equal amongst themselves.

11. Parallel Lines have a common Perpendicular: That is, the right Line which is perpendicular to one of them, is perpendicular also to the other.

Fig. 21. 12. The two perpendicular Lines (L O, Q I) intercept equal Parts of the Parallels, L I, O Q.

13. Two right Lines do not comprehend a Space; for unto this there are required Three at the least.

14. Two right Lines cannot have one common Segment; for that they cut one another only in a Point.

Of Propositions some propose something to be done, and are called *Problems*; in others we proceed no further than bare Contemplation, which therefore are named *Theorems*.

PROPOSITIONS.

THE requisite Citations are found in the Margin. When Propositions are cited, the first Number designs the Proposition; the Letter *I*, with the Number following, signifies the Book. As when you meet with (*per* 5. *I*. 3.) you must read it thus, (By the Fifth Proposition of the Third Book.) The Figure is always to be sought amongst the Figures of that Book in which we are then conversant. The rest of the Citations are easy to be understood.

The primary Affections of Triangles and Parallelograms are delivered in this Book. The more famous Propositions are, 32, 35, 37, 41, 44, 45, 47.

PROPOSITION I. Problem.

UPON a given right Line (*AB*) to make Fig. 23.
an Equilateral Triangle.

From the Center *A*, with the Interval (*AB*) (*a*) describe (*a*) *Per* Po- the Circle *FCB*; and from the Center *B* with the same *Int.* 3. Interval *BA* describe the Circle *ACL*, cutting the former in the Point *C*, from which Point draw the right Lines *CA*, *CB*.

I say, that the Triangle *ACB* now made, is Equilateral. For the right Line *AC* is equal to the right (*b*) Line *AB*, (*b*) *Per* seeing they are Semi-diameters of the same Circle *FCB*. *Def.* 18. And again, the right Line *BC* is equal to the same right Line *BA*, seeing they are both Semi-diameters of the Circle *LCA*. Therefore, *AC*, *BC* are (*c*) equal betwixt them- (*c*) *Per* selves. And therefore all the Sides of the Triangle are e- *Axiom.* 1. qual. Therefore the Triangle (*d*) *ACB* is both Equilate- (*d*) *Per* ral, and made upon the given Line *AB*; which was the *Def.* 25. thing to be done. *Q. E. F.*

Corollary. " Hence we may measure an inaccessible Fig. 77.
" Line, as *AB*. For suppose an Equilateral Triangle
" whatsoever *BDE* applied to the Point *B*, along the Line
" *BA*. Looking from the Point *B*, along the Line *BE*,
" mark as many Points as you conveniently can in the Line
" *BC*. Then remove the Triangle *BDE* along the Line
" *BC*, from one place to another of that Line, until, by
" taking

“ taking aim along the Side of the Triangle E D or C F,
 “ you see the inaccessible point A in a continuation of that
 “ Line. Thus the Triangle B A C is as well Equilateral
 “ as B D E. If therefore you shall now measure the ac-
 “ cessible Line B C, you have the measure of the inaccessi-
 “ ble A B. Q. E. F.

PROP. II. Problem.

Fig. 24. FROM a given Point A, to draw a right Line equal to one given, E F.

Take with a pair of compasses the Interval E F, and transfer it from A to D, the right Line A D will be equal to the given E F.

PROP. III. Problem.

Fig. 24. TWO unequal right Lines being given, from the greater of them G H to cut off G I equal to the less E F.

Take with a pair of Compasses the Interval of the less given Line E F, and transfer it unto the greater from G to I.

PROP. IV. Theorem.

Fig. 25. IF in two Triangles (X, Z) one side of the one (B A) be equal to one side (F L) of the other, and another side (C A) of the one equal to another side (I L) of the other, and the Angles (A and L) made by those sides be also equal; then the Bases (B C, F I) are likewise equal, as also the Angles at the Bases (B F and C I) which are opposite to equal sides, and consequently the whole Triangles are equal.

For

For if we suppose the Triangle Z to be laid upon the Triangle X, the sides L F, L I will perfectly agree and fall in together with the sides that are equal to them, A B, A C, and this in such sort (*c*) that the three Points (L. F. *c*) *Per* 1.) shall fall upon three Points, (A. B. C.) Therefore the whole Base F I will also fall upon the whole Base B C. But then the Angles F, B, and likewise those, I, C, and the whole Triangles will mutually (*congruere*) agree to each other. All therefore, by the 7th *Axiom*, are equal. *Q. E. D.* Which was the Thing to be demonstrated.

Corollary. “ (1.) Hence we may also in another way *Fig. 8.*
 “ measure the Line A B, although otherwise impracticable
 “ by reason of some Obstacle, as a River, &c. between
 “ the Extremities thereof. For from any point whatsoever, as the Point C, let the Angle A C B be observed,
 “ and then let the Lines A C, B C be measured; and in
 “ any accessible Plane let there be measured about the
 “ Angle F, which is equal to the Angle C, two Lines F D
 “ and F E, which are equal to the Lines A C and B C
 “ respectively. And there will be the accessible Line D E
 “ equal to the inaccessible A B, *Q. E. I.*

Corollary. (2.) “ Hence also those who play at Billiards with Ivory Balls, may learn how by the Reflexion *Fig. 79.*
 “ of their own to hit and remove their Adversary's Ball.
 “ For let B be the Ball to be stricken, A that which is to
 “ strike it, and C D the Rectilinear Plane. Let the Line
 “ B E be perpendicular to the Line C D, and D E be
 “ equal to D B. If the Ball A be stricken, and carried along the right side A F E unto the Point F, it will there
 “ be so reflected, that after the Reflexion it will tend unto
 “ B. For in the Triangles B F D, E F D, the side F D
 “ is common to both, and the side B D is equal to the
 “ Side D E; and the Angles at D are equal, as being
 “ right ones. The whole Triangles therefore are equal;
 “ and therefore the Angle B F D, which is equal to the
 “ Angle D F E, is * equal to A F C, the Angle A F C * *Per* 15.
 “ being vertically opposite to D F E. Wherefore, seeing *l. 1.*
 “ the Angle A F C is the Angle of Incidence, which in
 “ such Cases is equal to the Angle of Reflexion, it is manifest that B F D, which hath been proved equal to A F C,
 “ is the Angle of the Reflexion of the Ball A, and that
 “ the

" the Ball tending towards E, is in the Point F so reflected,
 " as to hit the Ball B. *Q. E. D.*

Scholium or Observation.

BY much the same Way of Reasoning, whereby this 4th *Proposition* has been demonstrated, the following *Theorem*, which we shall have occasion to use by and by, may be demonstrated also.

Fig. 25. If in two Triangles X, Z, the Sides B C and F I shall be equal, and the Angles adjacent to these two Sides equal also, *viz.* B and C equal to F and I; all the other Things, and the whole Triangles themselves will be equal.

For the Side F I laid upon the Side B C will agree, or thoroughly coincide with it (a.) And then because the Angles B and C are equal to those F and I, when the Side F I is laid upon the Side B C, F L (b) will fall exactly upon B A, and I L upon C A. Therefore the Point L will fall upon the Point A (for if it fall without A, the Sides F L, I L would not fall upon the Sides B A, C A.) Therefore all Things are equal, by the 7th *Axiom*.

(a) *Per*
Axiom 8.
 (b) *Per*
Axiom 8.

PROP. V. Theorem.

Fig. 26. **I**N an *Isocles*, or *Equicrural Triangle*, the Angles at the Base (A, C) are equal.

Let the Triangle A B C be understood to be twice put, but in an inverted posture *c b a*. Because therefore, in the two Triangles A B C, *c b a*, the Side A B is by the Supposition, equal to the Side *c b*, and the Side C B to the Side *a b*, and the Angle B to the Angle *b*; the Angle A also at the Base (*c*) will be equal to the Angle *c*. *Q. E. D.* For as *l. 1.* for the Angles C and *c* they are the same.

(c) *Per* 4.
l. 1.

Corollary.

THEREFORE an Equilateral Triangle is also Equiangular.

PROP.

PROP. VI. Theorem.

IF in a Triangle (ABC) two Angles (A and C) be equal, the Sides also (AB , BC) which are opposite to those Angles are equal also. Fig. 6.

Let the Triangle ABC be supposed to be twice put but in an inverse Situation, cba . Because therefore in the Triangles ABC , cba , one Side AC is equal to one Side (ca) and the Angle A is equal to the Angle c , and the Angle C equal to the Angle a , all the other Things shall be likewise (a) equal, and consequently AB shall be equal to ^{(a) Per} the Side cb . *Q. E. D.* For as for the Line CB and cb ^{Schol.} they are the same. Prop. 4.

Corollary.

THEREFORE an Equiangled Triangle, is also Equilateral.

Corollary (2.) " Hence, by the means of the shadow of Fig. 80.
 " the Sun, we may measure the height of a Tower, or
 " any elevated Point. For when the Sun is elevated 45
 " Degrees above the Horizon, the shadow which the
 " Tower casts towards the Horizon, will be exactly equal
 " to its Height. For, by reason that the Angle ACB is
 " half a right Angle, the Angle BAC also * will be half * Per Co-
 " a right one; and so by the Force of the present Propositio- rol. 11.
 " on, the Line AB will be exactly equal to the Line BC . Prop. 32.
 " The Line BC therefore being found by measuring, there l. 1.
 " is found at the same time the Line AB , the Height of
 " the Tower above the Horizon.

Corollary (3.) " The same Thing also may be found
 " without the Sun by the means of an Astronomical Qua-
 " drant. For where the Angle of Elevation is half-right,
 " there the Height of the tower above the Observer's Eye
 " is equal to the distance of the same Eye, from that part
 " of the Tower which is opposite to it. The distance there-
 " fore of the Eye from the Tower being given by measur-
 " ing, there is given at the same time the Height of the
 " Tower. *Q. E. I.*

The

The Seventh Proposition in *Euclid* is for the sake of the Eighth, which without it will here be demonstrated.

PROP. VIII. Theorem.

Fig. 27. **I**F two Triangles (X, Z) have all their Sides equal amongst themselves respectively (AC equal to EF ; CB to FI ; AB to EI ;) they will also have all the Angles which are opposite to equal Sides, equal: (C equal to F ; A to E ; B to I .)

For suppose the Side AB laid upon its Equal EI , if then the Point C falls upon F , the Triangles will in the Whole agree or coincide, and consequently all the Angles will be equal. But the Point C will fall upon the Point F . For,

Fig. 81. “ From the Center A let a Circle be described with the Semi-diameter EF ; and from the Center I , let another Circle be described with the Semi-diameter IF ; the Point C by reason of the Equality of the Sides of both Triangles, will be in the Circumference of both Circles, and consequently in the Point E , the common Intersection of both these Circumferences. *Q. E. D.*

PROP. IX. Problem.

Fig. 29. **T**O Bisect or Divide into two equal Parts, a given right lin'd Angle, as $I A L$.

From the Sides of the Angle take with a pair of compasses two equal Lines; AB, AC ; then from the Centers B and C describe two equal Circles cutting one another in F ; which done, draw the Line FA . This bisects the Angle.

For draw the Line BF, CF ; the Triangles FAB, FAC are to each other Equilateral; for the Sides AB, AC are by the Construction equal, as in like manner are the Sides BF, CF , they being Semi-diameters of equal Circles; and AF is common to both Triangles. Therefore

(d) *Per 8.* the Angles $B A F, C A F$ (d) are equal. Therefore the given Angle $I A L$ is bisected. *Q. E. F.*

Corollary.

Corollary.

HENCE we may learn how an Angle may be divided into all equal Angles, 4, 8, 16, &c. viz. by bisecting each part again.

Scholium.

NO one hath hitherto taught the Way of dividing Angles into all equal Parts whatsoever with a pair of Compasses and a Rule.

Yet may you divide any given Angle mechanically into any equal Parts whatsoever, if from the top of the Angle, as the Center, you describe an Arch between the Legs of the Angle, and divide the Arch into as many equal parts as you require; for right Lines let down from A, through the points of the Division, will cut the Angle into so many equal parts. Fig. 30.

PROP. X. Problem.

TO Bisect a finite given Line (*AB*.)

Fig. 31.

Upon the given Line *AB* make an Equilateral (*a*) Tri-
angle *AGB*, bisect its Angle *G* (*b*) with the right Line
GC. The same shall bisect the given Line *AB*. (a) Per 1.
(b) Per

For in the Triangles *X*, *Z*, the Side *CG* is common; praced.
and by the Construction *GB*, *GA* are equal and the
Angles contained between them *AGC*, *BGC*, are like-
wise equal. Therefore the Bases *AC*, *BC* (*c*) are equal. (c) Per 4.
The given Line therefore *AB* is bisected. Q. E. F. l. 1.

But for practice, it is sufficient, from the Centers *A* and *B*, to describe two equal Circles cutting one another in *G* and *L*, and so to draw the right Line *GL*.

PROP. XI. Problem.

FROM a given Point (*A*) to raise a Per-
pendicular in a given right Line *LI*. Fig. 32.

With a pair of Compasses take the equal Lines *AC*,
AF. From the Center *C* and *F* describe two Circles,
cutting

cutting one another in B. The Line which is drawn from B to A will be the Perpendicular required.

- For let the right Lines C B, F B be drawn. The Triangles X and Z are Equilateral to one another. Therefore
 (a) *Per* 8. the Angles C A B, F A B are equal (*a*). Therefore B A
l. 1. is (*b*) perpendicular to the Line (L I.) *Q. E. F.*
 (b) *Per* In Practice this and the next are easily performed by the
Def. 14. help of a Square.

PROP. XII. Problem.

Fig. 33. FROM a given Point (A) which is without an infinite right Line (as L Q) to let fall a Perpendicular to that Line.

- From the Center A describe a Circle which may cut the
 (c) *Per* 10. given L Q in C and I. Bisect the right Line C I (*c*) with
l. 1. the right Line A B. This A B is the Perpendicular required.

- For let there be drawn A C, A I. Because by the Construction, the Triangles X and Z are Equilateral to one
 (d) *Per* 8. another. Therefore the Angles (*d*) C B A, I B A, are
l. 1. equal. Therefore A B is (*e*) Perpendicular. *Q. E. F.*

- (e) *Per*
Def. 14.

PROP. XIII. Theorem.

Fig. 34. THE right Line (B A) standing upon the right Line (C F) either makes two right Angles, or Angles equal to two right ones.

- For if B A stand upon it perpendicularly, then by the 14th *Definition*, the two Angles B A C, B A F will be right ones. And if B A stand obliquely, let there be raised
 (f) *Per* (*f*) the Perpendicular A L. Where, because the unequal
 11. *l. 1.* Angles C A B, F A B possess the same place which the two right ones C A L, L A F do, and agree to them, they
 (g) *Per* are equal (*g*) to them. *Q. E. D.*

Axiom 7.

Corollaries.

1. IN the same manner it will be demonstrated if more right Lines than one stand upon the same right Line, that the Angles thereby made are equal to two right ones.

2. Two

2. Two right Lines cutting one another, BAC, FAL , *Fig. 37*. make the Angles equal to four right ones.
3. All the Angles which are about one Point, make *Fig. 36*. Angles equal to four right ones. It appears from *Corollary* 2. for they are four right ones, cut into more Parts.
4. The Angle CAF being known, you at the same time *Fig. 37*. know its Complement unto two right Angles BAF . For Example, Let the Angle CAF be of 70 Degrees; the Angle BAF will be of 110 Degrees. For those two Numbers added together make 180 Degrees, which is the Measure of two right Angles.

PROP. XIV. Theorem.

If two right Lines (XR, ZR) at the same Point of a right Line (QR) make the Angles on both Sides (XRQ, ZRQ) equal to two right Angles; the Lines (XR, ZR) make one right Line.

If you deny it, Let XR, BR make one right Line. (a) *Per 13*. Therefore the Angles XRQ, QRB (a) will make two right Angles. Which thing is (b) absurd; seeing by the Hypothesis XRQ, ZRQ do make two right Angles. *Axiom 9.*

PROP. XV. Theorem.

If two right Lines (BC, FL) cut one another in A , the Angles opposite at the top (A) are equal, viz. LAB to CAF , and BAF to LAC .

For because BA stands upon the right Line LF , the Angles LAB, FAB are (c) equal to two right ones: And (c) *Per 13*. because FA stands upon the right Line BC , the Angles FAC, FAB are also equal (d) to two right ones. Therefore the two Angles together (e) LAB, FAB are equal to those two together FAC, FAB ; by taking away FAB , (e) *Per* common to both, LAB (f) remains equal to FAC . In the same manner BAF, LAC are shewed to be equal. (f) *Per*

Corollary. " From these two Propositions we gather in Catoptrics, that a Ray of Light, as reflected in an Angle C equal

- Fig. 82. " equal to the Angle of Incidence, taketh the shortest way
 " of all. *e. g.* When the Angles BED , AEF are
 " equal, the Lines AE and EB taken together, are
 " shorter than any Lines whatsoever, as AF and FB taken
 " together. For from the Point B , let the perpendicular
 " Line BC be let down; and let BD and DC be equal;
 " Let the Lines also EC and FC be drawn. Now in the
 " Triangle BED and DEC , seeing the Side DE is com-
 " mon to both, and the Side BD and DC are equal by
 " the Hypothesis, as is also in the like manner BDE equal
 " to the Angle CDE ; the Triangles also shall be * equal
 " in all other Things, and BE shall be equal to CE , and
 " the Angle BED to the Angle DEC ; (where, because
 " the Angle DEC is equal to $[BED, \text{ that is}] AEF$, the
 " Lines AE , EC are proved to make one right Line.)
 " And in the same manner the Line BF will be proved
 " equal to FC . Seeing therefore the Lines BE and EA
 " taken together; are equal to the Line CA , and the Lines
 " BF , FA taken together, are equal to the Lines CF , FA
 " taken together; it is manifest that CA , which is one
 " Side of the Triangle ACF * is less than the two Sides
 " CF , FA taken together. *Q. E. D.*
- * Per 4.
 l. 1.
- * Per 20.
 l. 1.

PROP. XVI, XVII.

THESE two Propositions are contained in
 Proposition 32; and are not here made use
 of till then.

PROP. XVIII. Theorem.

- Fig. 38. **I**N every Triangle the Angle (A) which is op-
 posed to the greater Side (BO) is the greater
 and (B) which is opposite to the lesser Side (AO)
 is the lesser Angle.

- (A) cannot be equal to (B) for then the opposite Sides
 (a) Per 6. BO , AO would be equal (a); which is contrary to the
 l. 1. Hypothesis. Neither can A be less than B , for if it were
 so, there might within the Angle B be made an Angle ABF
 by the right Line BF ; which Angle should be equal to A .
 But then by the 6th of this Book BF , AF shall be equal
 and

and if you add to both OF , then BF , FO shall be equal to AO . But AO by the Hypothesis is less than BO . Therefore BF , FO shall be less than BO , which contradicts the Definition of a right Line, which is the shortest of all betwixt two Points. Therefore the Angle A is neither less than B , nor equal to it. Therefore it is greater. *Q. E. D.*

PROP. XIX. Theorem.

IN the Triangle AOB the Side (BO) which is opposed to the greater Angle (A) is the greater; And (AO) which is opposed to the lesser Angle B , is the lesser. Fig. 38.

This Proposition is the converse of the former. BO is not less than AO , for if it were, the Angle (A) by the 8th would be less than B ; which is contrary to the Hypothesis. Nor can BO be equal to AO , for in this case, by the 5th, the Angles A and B would be equal. But this Equality of those Angles is contrary to the Hypothesis. Therefore BO is greater than AO . *Q. E. D.*

Corol. "Hence we gather, that a Globe, or Ball perfectly polished, cannot rest in an horizontal Plane perfectly polished, but where it toucheth the Earth. For let the Line AB be an horizontal Plane, C the Earth's Centre, CA the Semi-diameter of the Earth, perpendicular to the Tangent AB . The Globe placed at B , because of its Gravity, and the Declivity of the Plane, will descend towards A . For in the Triangle CAB , the perpendicular Line CA , which is opposite to the acute Angle ABC , is less than the Line BC , which is opposed to the right Angle BAC ; and so there is from B to A a perpetual Descent, in which the Globe cannot rest. And in the like manner we prove the Descent of Fluids and their Conformation into a spherical Surface." Fig. 83.

PROP. XX. Theorem.

IN any Triangle, any two Sides of it taken together, are greater than the remaining Side.

This, with *Archimedes*, is, as it were an *Axiom*; forasmuch as it is immediately manifest out of his Definition of a right Line; which see above amongst the Definitions.

PROP. XXI. Theorem.

Fig. 39. **I**F from the Ends of one Side AB , two right Lines be drawn, and joined together within the Triangle, (as the Lines AO , BO) these are less than the Sides of the Triangle (AC , BC) but they comprehend a greater Angle (AOB .)

For as for the first Part of the Proposition, draw out (a) *Per 20.* AO unto F : AC , CF are (a) greater than AF . Therefore the common Line FB being added, AC , BC are (b) By the same. greater than AF , FB . Again, OF , FB are greater than OB . Therefore the common AO being added, AF , BF are greater than AO , BO . Therefore AC , CB are much greater than AO , OB .

The second Part of this Proposition will be demonstrated in the second Corollary of the first Part of Proposition 32. And in the mean while we shall make no use of it.

PROP. XXII. Problem.

Fig. 40. **T**O make a Triangle of three given right Lines (BO , LB , LO) of which any two must be greater than the third.

Let BL , one of the given Lines be taken, and B one of its Extremities being taken for the Center, with the Interval of the other given Line BO , describe an Arch.

Then the other Extremity L being taken for the Center, with the Interval of the third given Line LO , describe an Arch, cutting the former in O ; which being done, and the right Lines BO , LO being drawn, I say that that is done, which was to be done.

The Demonstration is manifest from the Construction.

PROP. XXIII. Problem.

AT a given Point in a right Line (as B) to make an Angle equal to a given one (A.)

First of all let CF be drawn at a venture, cutting the Sides of the given Angle A. Then in the given right Line from B, take BL equal to AF. Then from the Center B describe a Circle with the Interval AC; afterwards another from the Center L, with the Interval FC, which may cut the former in O. Then from O unto B, and L having drawn right Lines, the Angle LBO will be equal to the given one A. Fig. 40.

For by the Construction, the Triangles are Equilateral to one another. Therefore by the 8th of this Book the Angles B and A are equal.

Scholium.

IT seems meet for the sake of Beginners to propound some things here which are necessary for Practice about Angles.

The Measure of an Angle is the Arch of a Circle, which is described from A, the Top of the Angle as the Center. Fig. 41. Therefore look how many Degrees the Arch BC, which is intercepted between the Legs of the Angle BAC shall contain, of so many Degrees the Angle BAC shall be said to be. And so because BF, a quarter of the Circumference, contains 90 Degrees, and measures the right Angle BAF, a right Angle shall be said to be of 90 Degrees. In like manner, because half the Circumference, which is divided into 180 Degrees, measures two right Angles, and the whole Circumference, which is divided into 360 Degrees, measures four right Angles; two right Angles shall be said to make 180 Degrees, and four 360 Degrees. These Things being premised, the Practice about Angles is as follows.

1. At B a given Point in a right Line to make an Angle equal to the given one A. Fig. 42.

From A, the Top of the given Angle as the Center, describe betwixt the Sides of the Arch CF. Then from B, the given Point as the Center, describe with the same Interval the Arch LZ; from which take off LO equal to CF. Through B and O draw a right Line; LBO shall be equal to the given A.

Fig. 3.

2. To examine the Degrees of the given Angle OPQ . This is done very easily through any Semi-circle or Protractor, which is divided into 180 Degrees. For put the Center of the Semi-circle upon P , the top of the Angle, and the Radius of the Semi-circle PL upon the Side of the Angle PQ ; and the Arch LO , which is intercepted betwixt the Legs of the Angle, will shew of how many Degrees the given Angle is.

3. To frame an Angle, containing a given Number of Degrees, as 42.

Fig. 43.

Draw the right Line XQ , in which mark the Point P . Upon P put the Center of a Semi-circle, and its Semi-diameter PL upon PQ . From L number 42 Degrees, that is, until you come to O . A right Line drawn from P through O , will give the Angle OPL of 42 Degrees.

PROP. XXIV, XXV. Theorems.

Fig. 44.

If two Triangles (BAC , BAF ,) shall have two Sides (BA , AC ,) equal to two (BA , AF ,) one Side of one, to one Side of the other; and if one of the Triangles bath the Angle (BAF) contained by those Sides greater than the other (BAC ,) it shall have the Base BF greater than the Base (BC .)

And again, if it bath the Base greater, it shall have the Angle greater.

From the Center A , describe a Circle which passeth through C , it shall pass also through F , because AC , AF are supposed to be equal. Therefore BF shall fall betwixt the Point A and C . Then join CF . The Angle BCF is greater than the Angle ACF ; that is, by the 5th of this Book, than the Angle AFC , and consequently much greater than the Angle BFC . Therefore in the Triangle BCF ,

(a) Per 19.
L. 1.

(a) BF , which is opposite to the greater Angle BCF , is greater than BC , which is opposite to the lesser Angle BFC .

2. As for the second Part of the Proposition, this is manifest from the first Part, and Proposition 4.

PROP. XXVI. Theorem.

IF two Triangles (*X* and *Z*) have two Angles Fig. 25.
equal to two, one Angle of the one, equal to one
Angle of the other (*B* to *F* and *C* to *I*), and one
Side of one Equal to one of the other, whether
it be that which is betwixt the equal Angles
(as *BC*=*FI*) or a Side which is opposed to one
of the equal Angles (as *AC*=*LI*), all the other
Parts shall be equal.

For in the Place, let the Sides (*BC*, *FI*) which are be-
twixt the equal Angles, be supposed equal: In this Case
all the other Parts are equal: as hath been already demon-
strated in the *Scholium* of the 4th Proposition.

Again, suppose the Sides *AC*, *LI*, which are opposed
to the equal Angles to be equal. Here, because the Angles
(*B*, *C*) are by the Hypothesis equal to (*F*, *I*) the other
Angles, also (*A*, *L*) shall be equal by Corollary 9. Proposi-
tion 32. which Proposition depends not upon this. There-
fore by the first Part of this, all the other Parts are equal.

Corollary. " Hence also, following *Thales*, we may Fig. 84.
" measure inaccessible Distances. *e. g.* Let *AD* be an
" inaccessible Line; to which at the Point *A*, let there be
" erected the Perpendicular *AC*. Let there be made the
" Angle (*ACB*) equal to the Angle (*ACD*) the accessible
" Line *AB* shall be equal to the inaccessible *AD*. Q. E. I.

PROP. XXVII. Theorem.

IF the right Line *GO* shall cut two right Lines Fig. 45.
which are parallel (*AB*, *CF*), 1. The alter-
nate Angles (*RO*, *LO*, likewise *BL*, *OL*) shall be equal. 2. The external Angle
GLB shall be equal to the internal one on the
same Side (that is, to *LOF*) as likewise *GLR*
equal to *LOC*. 3. The two internal ones on the
same

same Side ($AL O, CO L$) as taken together, shall be equal to two right ones, as likewise the two ($BL O, FO L$) equal to two right ones.

Fig. 46. The first Part is thus proved. From O and L draw the Perpendiculars OR, LQ . These are perpendicular to the

* *Per* * two Parallels AB, CF ; and by Definition 36, equal betwixt themselves, they shall therefore (a) intercept equal

Axiom 11. (a) *Per* Parts of the Parallels, and RL shall be equal to QO .

Axiom 12. Therefore the Triangles X and Z are Equilateral to one

(b) *Per* 8. another. Therefore (b) the alternate Angles $RL O, QOL$, which are opposite to the equal Sides RO, QL , are equal.

l. 1. Which is the first Thing. From whence it is likewise manifest, that the Alternates $BL O, CO L$ are equal. For

because, as well $BL O, AL O$, as $CO L, FO L$ are equal

(c) *Per* 13. (c) to two right ones; therefore $BL O, AL O$ together,

l. 1. are equal to $CO L, FO L$. Therefore taking away the Equals $RL O, FO L$, the remaining ones $BL O, CO L$, shall be likewise equal.

Part second. The Angle GLB is equal to that which (d) *Per* 15. is vertically opposite $RL O$ (d). But $RL O$, by the first

l. 1. Part of this Proposition, is equal to LOF . Therefore GLB , the external Angle, is equal to the internal remote one, which is on the same Side, LOF .

Part third. ALO , by the first Part, is equal to LOF .

(e) *Per* 13. But LOF , with COL , make (e) Angles equal to two right ones. Therefore ALO , with COL , doth the same.

l. 1.

Fig. 85. Corol. "Hence in Imitation of *Eratoſthenes*, we learn
"to measure the Compass of the Earth. For he observed,
"that on the Day of the Summer Solstice, the Sun was
"perpendicularly over *Siene*, a City of *Egypt*; and he
"found by the means of a Stile, perpendicularly erected,
"that on the same Day the Sun was distant from the vertical Point of *Alexandria*, a City of *Egypt*, situate almost
"under the same Meridian with the other, seven Degrees,
"with one Fifth Part of a Degree; and he knew that
"these two Cities were about 5000 Furlongs distant from
"each other. From these Things, by the help of this
"Proposition, he determin'd the Compass of the Earth.
"Let A be *Siene*, and B be *Alexandria*, where the Gnomon
" BC is erected perpendicular to the Horizon. Let DF

"and

“ and E G be the Solar Ray's parallel to one another as to
 “ Sense. D A a Ray perpendicular to the Horizon of *Si-*
 “ *ene*; and E G a Ray oblique to the Horizon of *Alex-*
 “ *andria*, and which passing by the top of the Gnomon,
 “ makes with it the Angle G C F, which is of $7\frac{1}{2}$ Degrees.
 “ Now seeing the Angle G C F, is equal to the alternate
 “ one A F B, and the measure of it is the Arch A B of
 “ $7\frac{1}{2}$ Degrees; he found the Compass of the Earth by
 “ this Analogy, as $7\frac{1}{2}$ Degrees are to 5000 Furlongs; so
 “ the whole Circumference, which is of 360 Degrees, is
 “ in a gross Number to 250000, the Compass of the Earth
 “ in the same measure. Q E. I.

PROP. XXVIII. Theorem.

If a right Line (G O) cutting two right Lines Fig. 47.
 (A B, C F) makes the alternate Angles
 (A L O, L O F) equal; the Lines (A B,
 P C) are parallel.

If you deny it, let X L Z, passing through the Point
 L, be parallel to C F. Therefore X L O (a) is equal to (a) By the
 the alternate F O L, which cannot be, seeing by the Hypo-foregoing.
 thesis A L O is equal to F O L.

PROP. XXIX. Theorem.

If a right Line (G O) cutting two right Lines Fig. 45, 46.
 (A B, C F) shall make the external Angle
 (G L B) equal to the internal opposite one
 (L O F); or shall make the two internal Angles on
 the same Side (A L O, C O L) equal to two right
 Angles; (A B, C F) are parallel Lines.

By the 15th of this Book, G L B is equal to A L O,
 which is vertically opposite to it. But by the Hypothesis
 G L B is equal to L O F. Therefore also A L O is equal to (b) By the
 its alternate one L O F. Therefore (b) A B, C F are parallel, foregoing.

Again, C O L with F O L make Angles equal to two
 right ones. But by the Hypothesis C O L with A L O,
 make in all two right Angles also. Therefore A L O,
 F O L,

F O L, the alternate Angles are equal. Therefore again,
 (a) *By the* (a) A B, C F are parallel.
foregoing.

Corollary. " From the second Part of this Proposition
 " it appears that every Rectangle is a Parallelogram.

P R O P. XXX. Theorem.

Fig. 45. **I**F two right Lines (A B, C F) be parallel to
 the same right Line (D N) they are parallel
 betwixt themselves.

It is manifest in itself, and from the foregoing Proposi-
 tions. For if all be cut by the right Line G O, the exter-
 (b) *Per 27.* nal Angle G L B is equal (b) to the internal opposite one
 l. 1. L D N. Now L D N is an external Angle in respect of
 (c) *By the* D O F, and therefore (c) equal to it. Therefore also G L B
same. is equal to L O F. Therefore A B, C F (d) are parallel.
 (d) *By the* foregoing.

P R O P. XXXI. Problem.

Fig. 48. **T**HROUGH a given Point (A) to draw a
 Parallel to a given right Line (F C).

From the Point A, let there be drawn at random A L,
 cutting the given F C. At the Point A, let there be made
 (e) *Per 23.* the Angle (e) L A S equal to the Angle A L F. The Line
 l. 1. A S will be parallel to C F; as is manifest from the 28th,
 the alternate Angles S A L, A L F being equal.

As for the Practice. Draw A L, and from the Center
 L describe an Arch I Q; and from the Center A, with the
 same interval, describe the Arch O X; from which, hav-
 ing taken off O B equal to I Q, the right line drawn
 through A and B will be the Parallel sought. The Demon-
 stration depends upon the 29th, l. 1.

Fig. 49. Or otherwise thus. From a certain Center P describe a
 Circle which may pass through the given Point A, and
 may cut the given Line C F in Q and O. Take the Arch
 O N equal to Q A. The right Line A N shall be the Pa-
 rallel sought.

The Demonstration hereof depends upon the 29th, l. 3.
 and the 28th of this.

P R O P.

PROP. XXXII. Theorem.

PART I.

IN every Triangle any one of the external Angles (as $\angle FBC$) is equal to the internal remote ones (A and C). Fig. 51.

Through the Point B draw (a) BL parallel to AC : (a) *Per* 31. Because FA cuts the two Parallels BL , AC , the external $\angle FBL$ shall be equal to the internal one A (b) And (b) *Per* 27. because the Line BC cuts the same Parallels (BL , AC). $\angle LBC$ shall be (c) equal to its alternate one C . (c) *By the* Therefore the whole Angle FBC shall be equal to A and C both together. $\angle$ *E. D.*

Corollaries.

1. **T**HE external Angle FBC is greater than either of the internal opposite ones A or C . Fig. 51.
2. Of the Angles (C and AOB) having the same Base, AB which falls within, is the greater. Fig. 39.

For let AO be produced unto F , AOB , by this Proposition, is greater than OFB ; and likewise OFB is by this greater than C . Therefore OFB is much greater than C . Fig. 55.

3. If from one Point A there falls two right Lines upon BC ; one of them AO obliquely, the other AF perpendicularly; this last shall fall on the Side of the acute Angle AOB . For let it fall, if it may be, on the Side of the obtuse Angle AOC ; as for instance, in Q . In this Case, the acute Angle AOB shall be external in respect of AQB , and consequently shall be greater than the right one, by *Corollary 1.* which is absurd.

PROP.

PROP. XXXII. Theorem.

PART. II.

IN every Triangle the three Angles taken together, are equal to two right ones, and therefore make 180 Degrees.

Fig. 52.

(a) By the first Part Draw forth one Side A B unto F. The external Angle F B C is equal (a) to the two internal opposite ones, A and C. But F B C with A B C, make (b) Angles equal to two right ones. Therefore the two, A and C, with the same C B A, make Angles equal to two right ones. Q. E. D.

Fig. 53.

Or thus. Draw the Line H M parallel to A C, the alternate Angles, as well O and A, as N and C, (c) are equal. But O, Q, N, make Angles (d) equal to two right ones. Therefore also A, C, Q are equal to two right ones. Q. E. D.

Prop. 13.

l. 1.

Corollaries.

4. **T**HE three Angles of any one Triangle taken together, are equal to the three Angles of any other Triangle taken together.

5. If in a Triangle one Angle be right (or obtuse) the rest are acute.

6. If in a Triangle one Angle be right, the two other Angles together make one right Angle.

7. In every Triangle, the Angle which is right, is equal to the other two taken together.

8. When you know of how many Degrees one Angle of a Triangle is, you know at the same time how many Degrees the two other Angles, as taken together, do make up. And so on the contrary, when you know how many Degrees two Angles of a Triangle taken together do make up, or what is the Sum of them, you know at the same Time of how many Degrees the third Angle is.

9. When two Angles of one Triangle, either severally or together, are equal to two Angles of another Triangle; the third Angle of one Triangle is also equal to the third of the other.

10. When two Triangles have one equal Angle, the Sum also of the rest of the Angles are equal.

11. When

11. When in an *Isoceles*, the Angle contained by the equal Sides is a right one, the two other are, each of them, half-right Angles. And the Angles of an *Isoceles*, which are at the Base, are always acute.

12. In an Equilateral Triangle, each Angle is two thirds of a right Angle. For it is one third of two right ones, therefore it is two thirds of one right one.

13. Hence a right Angle (BAC) is easily divided into three equal parts; if upon AC be made the equilateral Triangle Z ; for seeing FAC is two thirds of one right one, BAF shall be one third of a right one. *Fig. 54.*

14. The Perpendicular AF is the shortest of all Lines which can be drawn from the Point (A) unto some right Line. For seeing the Angle F is a right one, AOF shall by the 5th *Corollary*, be an acute one. Therefore (*a*) AF is shorter than any other, as AO . *Fig. 55.* *Per 19. l. 1.*

15. Only one Perpendicular can fall from one Point unto one right Line. This is manifest out of the foregoing *Corollary*.

16. " Hence also we learn to determine the Parallax of the Stars, or the difference of their true and apparent Place. Let A be the Center of the Earth, B the Place of the Observer upon the Surface of the same. Let DBC be the Angle of the Star C , according to Observation, or the *visible* Angular distance thereof from the vertical Point; when in the mean while DAC is the *true* Angular Distance. Now the external Angle DBC , which is given from Observation, is equal to the Angles BAC and BCA , taken together; and consequently the Angle BCA is the difference of the Angles DBC and DAC . If therefore we shall from Astronomical Tables seek the Angle DAC , or what at that Time of Observation is the true Angular Distance of the Star from the vertical Point, when the Angle DBC is at the same Time known by means of the Quadrant, the Difference of those Angles BCA , which we call the *Parallax*, will likewise be known. *Q. E. I.* *Fig. 86.*

Scholium.

BY the Testimony of *Eudemus*, an ancient Geometrician, *Pythagoras* was the Finder out of this Proposition, which indeed is a Theorem most excellent in itself,

self, most fruitful in its Consecrations, and of immense use in all Parts of the Mathematicks. *Aristotle* very frequently makes mention of it, who also puts it for an Example of the most perfect Demonstration. But like, as from this Proposition, we have already learned, how many right Angles, the Angles of a Triangle are equivalent to; so by the help of the same, it will in the three following Propositions be manifest, how many right Angles, the Angles of any Rectilinear Figure whatsoever, whether internal or external do make.

Theorem I.

Fig. 56. **I**N every Quadrangular Figure, the four Angles together make four right ones.

For if, through the opposite Angles, you draw the right Line *B F*, this will cut the Quadrangle into two Triangles, without forming any new Angles, whose Angles together

(a) *Per 32.* do (a) make four right Angles.

l. 1.

Theorem 2.

AL L the Angles together of every right-lin'd Figure make twice so many right ones, abating four, as are the sides of the Figure.

Fig. 57. From any Point *A* within the Figure, let there be drawn unto the Angles of the Figure right Lines, which shall cut the Figure into so many Triangles as it hath Sides, and make no more Angles, but those of the Center. Wherefore, when each of the Triangles contains two right Angles

(b) *Per 32.* (b), they must all together contain twice so many right Angles as there are Sides. Now the Angles about the Point

(c) *Corol. 3.* *A* (c), do make four right Angles. Therefore, if from *Prop. 13.* the Angles of all the Triangles, you take away the new

l. 1. Angles which are about *A*, the remaining Angles, which indeed do alone constitute the Angles of the Figure, will make twice so many right Angles, excepting four, as are the Sides of the Figure.

Hence it appears, that all Right-lin'd Figures of the same Species, or Number of Sides and Angles, have the Sum of their Angles equal. Which thing is worthy of admiration.

The Practice is thus; Double the Denominator of the Figure, and from the Product take away four; the Remainder

mainder is the Number of the right Angles, which the internal Angles of the Figure do make.

Theorem 3.

AL L the external Angles of any Right-lin'd Figure *Fig. 58.* whatsoever taken together do make up four right Angles.

For each of the internal Angles of the Figure does $(d)(d)$ *Per 13.* with its respective external one, make two right Angles. *l. 1.* Therefore all the internal ones, together with all the external ones, do make up twice so many right Angles as are the Sides of the Figure. Now, by the Preceding, the internal ones, together with four right Angles added to them, make twice so many right Angles as are the Sides of the Figure. Therefore the external Angles are equal to four right ones.

Wonderful truly is this Property of Right-lin'd Figures; from whence it follows also, that all the Right-lin'd Figures of any Species whatsoever have the Sums of their external Angles equal. And therefore the three external Angles of a Triangle are equal to the thousand external Angles of a thousand-sided Figure. Which Observation is altogether worthy of Admiration.

P R O P. XXXIII. Theorem.

IF two right Lines, which are equal and parallel, as $(A B, C F)$ be joined by two others, $(A C, B F;)$ these will also be equal and parallel. *Fig. 59.*

Let $A F$ cut the Parallels $A B, C F$. In the Triangles $Q R$, the alternate Angles $B A F, C F A$ (a) will be equal. (a) *Per* Now the Side $A B$ is supposed equal to the Side $C F$, and 27. *l. 1.* $A F$ is common to both Triangles. Therefore (b) the Bases (b) *Per* $B F, A C$ are equal. (Which is the first Part.) And then 4. *l. 1.* the Angles at the Bases $A F B, F A C$ are equal; which being made by $A F$ falling upon the right Lines $A C$ and $B F$, are alternate Angles $A F B, F A C$ equal. Therefore $A C, B F$ are also (c) parallel. Which is the other Part. (c) *Per* 28. *l. 1.*

Corollary.

Fig. 87.

Corollary. 1. "Hence we learn to measure as well
 "the Heights of Mountains above the Horizon as their
 "horizontal Lines. Let ABC be the Side of a Mountain,
 "to which apply a great Square, or some Instrument
 "equivalent thereto ADB . Then shall AD be equal to
 " HB , and DB equal to AH . Then coming unto the
 "lower Part, which is from the Point B unto the Point C ,
 "practise as before. So shall EB be equal to CF , and
 " EC be equal to BF . Which done, the Sides parallel
 "to the Horizon, AD , BE , &c. added together will
 "give the horizontal Line GC ; and the perpendicular Sides
 " BD , EC , &c. added together, will give the Height AG .

Fig. 59.

Corollary. (2.) "Hence alio we learn to estimate the
 "Composition of Motions. Let a Body placed at A be
 "driven in the same Moment of Time by the Force AC ,
 "according to the Direction of the Line AC , and by the
 "Force AB , according to the Direction of the Line AB .
 "From the Conjunction of these two Forces it will de-
 "scribe the Diagonal AF . For in this Line of its Mo-
 "tion, neither of the Forces is changed: For the Body at
 " F is equally distant from both the Lines of Direction
 " AC , AB , as if it had been driven by either of the
 "Forces separately; which thing can be said of no other
 "Point. And this Corollary doth so fully agree with
 "Astronomical and other Mechanical Phænomena, that it
 "is worthily reckoned by the famous Sir *Isaac Newton*,
 "as a Foundation of his Geometrical Philosophy.

PROP. XXXIV. Theorem.

Fig. 59.

IN every Parallelogram the opposite Sides and
 Angles are equal, and it is cut into two equal
 Parts by the Diameter.

(a) *Per* Because AB , CF are (a) parallel, and AF falls upon
Def. 35. them, the alternate Angles BAF , CFA are (b) equal.
 (b) *Per 27.* Likewise because AC , BF (c) are parallel, and upon
l. 1. them falls the Line AF , the Alternates CAF , BFA (d)
 (c) *Per* are equal. Therefore the whole Angle BAC is equal to
Def. 35. the whole Angle $BF C$. In the same manner B and C are
 (d) *Per 27.* shewed to be equal. Which was the first Part.

l. 1. Now because it hath been already shewed, that the
 Triangles Q , R , which have one common Side AF ,
 have

have also the Angles adjacent to the common Side equal, $\angle B A F$ to $\angle C F A$; and $\angle C A F$ to $\angle B F A$; the Sides likewise shall be equal, (a) $A B$ to $F C$, and $B F$ to $A C$; and thus (a) *Per 26.* the whole Triangles are equal. Which was the second Part. *l. 1.*

Scholium.

PARTLY from this Theorem, and partly from a Definition to be premised to the second Book, the measuring *Fig. 60.* of a right-angled parallelogram is easily deduced. The Area thereof being produced by the Multiplication of the two contiguous Sides $A F$, $A C$ one by another. *e. g.* Let $A F$ be a Line of 8, $A C$ a Line of 4 Feet. Multiply 8 by 4, there arises 32 Square Feet for the Area of the Rectangle.

But the Area of a Square is had from the Multiplication *Fig. 61.* of the Side $F I$ by itself; as if $F I$ be of 5 Feet, multiply 5 into its self, there will arise 25 Square Feet for the Area of the Square.

The Demonstration is manifest from this Proposition, if parallel Lines be drawn through the Divisions of the Sides.

Corollary. "Hence Surveyors do easily divide the Area *Fig. 88.* of a Field, when it is a Parallelogram. For let $A B C D$ be the Parallelogram Field: $A D$ the Diameter, or Diagonal Line of the same, the middle Point whereof is marked F . Whatsoever right Line, as $E G$, passeth through the Point F , it divides the Field into equal Parts $E A C G$, $E B D G$. For the Triangle $A B D$ is equal to the Triangle $A C D$, and * the Triangle $A E F$ * *Per 26.* equal to the Triangle $G F D$. If therefore to the Trapezium $E B D F$, instead of the Triangle $A E F$, you shall add the Triangle which is equal to it, $G F D$, you will not change the Area; but the Trapezium $E B D G$ will be equal to the Triangle $A B D$, or to half the Parallelogram, and consequently to the Trapezium $A E G C$. *Q. E. D.*

PROP. XXXV, XXXVI. Theorems.

Fig. 62. **P**arallelograms upon the same or equal Bases (AB) and between the same Parallels (CQ, AX) are equal.

(a) *Per.* Because AL, BQ (a) are parallel, and CQ cuts them, the external Angle CLA shall (b) be equal to the internal one FQB . Then because, as well CF as LQ are equal (c) to the same AB , CF is equal to LQ . Add then FL to both, the whole Lines CL, FQ are equal. Moreover AL, BQ are equal (d). Therefore the Triangles CLA, FQB (e) are equal. Therefore taking away the common Triangle FOI , the Planes $FOAC, QBOI$ remain equal: To each of which Trapeziums add the Triangle AOB , the whole Parallelograms $ACFB, ALQB$ become equal. *Q. E. D.*

This Proposition will be made universal. *Prop. I. I. 6.* Beginners may here observe, that although, of two Parallelograms which are between the same Parallels infinitely produced and upon the same Base, one of them be extended unto an infinite Length, it still remains but equal to the other, by the Force of the present Demonstration.

[“ From hence it follows, that two Cities in Magnitude equal, may so much differ in Compass, that the Circumference of one may exceed that of the other an hundred or a thousand times. If, for instance, one be of a square Figure or Rectangular; but the other a Parallelogram, betwixt the same Parallels indeed with the former, but very oblong.

“ Moreover, it hence follows, that Figures of equal Compass round may contain Areas vastly different.]

Scholium.

Fig. 62. **F**ROM this Theorem we may learn to measure any Parallelogram. For the Area of it is produced from the perpendicular Altitude QX , or CA multiplied into the Base AB .

For

For the Area of the Rectangle C B which is equal to that of the Parallelogram B L is made (a) by A C, multiplying A B. Therefore, &c. (a) *By the foregoing Scholium.*

PROP. XXXVII, XXXVIII. Theorems.

Triangles (A C B, A L B) upon the same, or equal Bases (A B,) and between the same Parallels (C I, A Z) are equal. *Fig. 63.*

Draw the Lines B F, B I parallel to the Sides A C, A L. The Parallelograms A C F B, A L I B (b) are equal. But (b) *By the given Triangles are halves of those Parallelograms (c.) foregoing.* Therefore the given Triangles (d) are equal. (c) *Per 34.*

This Proposition will be made universal, *Prop. 1. 1. 1. 1.* Let Beginners mark the same Thing here concerning Triangles, which we bid them to note in the foregoing Proposition concerning Parallelograms. (d) *Per Axiom 6.*

Corollary (1.) "Hence Surveyors easily divide the Area of a Triangular Field. Let A B C be the Field, and let the Base B C be bisected in D. The Triangles A B D, A D C upon the equal Bases B D and D C, and having a common top A, or being between the same Parallels, are equal. *Q E. F.* *Fig. 89.*

Corollary (2.) "Hence we also gather, with the famous Sir *Isaac Newton*, that the Areas which all Bodies whatsoever that revolve round about an immoveable Center, towards which they are impell'd, do describe, are both in immoveable Planes, and are proportional to the Times of Description. For let the Time be divided into equal Parts; and in the first equal Part of Time, let the Body by the impress'd Force, describe the right Line A B. The same Body, in the second Part of Time, if nothing hindred, would go forward strait unto c, describing the Line B c, equal to A B; so that the Areas made by Lines drawn from the Center A S B, B S c (a) would be equal. But when the Body comes unto B, let the Force act with one single Impulse, but a great one, and make the Body to deflect from B c, and to go forwards in the right Line B C: i. e. let the centripetal Force be in that Place, to the Force before impuls'd, as C c or B g D 2. (a) *Per 37. l. 1.* " is

- (b) *Per* " is to Bc ; in this Case the Body will (*b*) describe the
Corol. 2. " Diagonal BC . Let there be drawn parallel to BS , the
Prop. 33. " right Line Cc meeting BC in C . In the second Part of
l. 1. " Time completed, the Body will be found in the Point C
 " in the same Plane with the first Triangle SAB . Join
 " SC . The Area made by a Ray drawn from the Center
 (c) *Per* 37. " that is, the Triangle SBc will be equal to (c) SBc , and
l. 1. " consequently to the first Triangle SAB (*d*.) By the
 (d) *Per* " same Argument, the Body, in the third equal Part of
Axiom 1. " Time, would, by its present Force, reach from C unto
 " d , so that the Line Cd should be equal to the Line Bc
 " or AB . But if the centripetal Force, whether it be
 " greater or less, does again act upon it in the Point C
 " in the end of the third Part of Time, it will be found
 " somewhere in the Line Dd , parallel to SC ; and there-
 " fore, as before, supposing the said Force to be equal or
 " unequal to what it was before, it will be found to have
 " described the Diagonal CD , and will be found in the
 " Point D ; and a Ray being drawn from the Center, the
 " Triangle SDC will be equal to that SdC , and conse-
 " quently to the others SCB , SAB , which are equal one
 " to the other. In like manner, if the centripetal Force
 " act successively in the Points D , E , F , and be the cause
 " that the Body, in the several Parts of Time respect-
 " ively, describe the Diagonals DE , EF , &c. the Area
 " now made, as a-fore, will be in the same Plane, and the
 " angles will be described equal to the former Triangles.
 " Therefore in equal Times, equal Areas are described in
 " an immoveable Plane; and so the sums of the Areas
 " $SADS$, $SAFS$ will be amongst themselves, as the
 " Times wherein they were described. Now let the Num-
 " ber of the Triangles be increased, and their Wideness di-
 " minished infinitely; both that last Perimeter of them
 " $ABCDE F$, will be a curve Line, and the Areas de-
 " scribed in one and the same immoveable Plane, will in
 " this Case also be proportional to the Times as well as be-
 " fore. *Q. E. D.*

PROP. XXXIX, XL. Theorems.

EQUAL Triangles (ACB, AFB) upon the same, or an equal Base (AB) and on the same Side, are between the same Parallels (AB, CF .) Fig. 64.

If you deny it, let CL be parallel to AB , and let BL be drawn. Then, ALB is equal to ACB (a). But by (a) *By the Hypothesis*, AFB is equal to ACB . Therefore *foregoing*. ALB and AFB are equal; *i. e.* a part is equal to the Whole. Which cannot be. Therefore, &c.

[Corollary (1.) "Hence also, with the famous Sir *Isaac Newton*, we gather, that all Bodies which are moved in Curve Lines, and describe Areas about some Center proportional to the Times, are perpetually urg'd and press'd by a Force impelling towards the Center. For because of the Equality of the Triangles SCB, ScB described upon the same Base SB , the Points C and c shall be in a Line Cc , which is parallel to the Base; and so the Figure $BcCg$ shall be a Parallelogram: the Sides whereof Bc and Bg are * the Lines of the Directions of * *Per Corol. 2.* the Forces, and BC is the Diagonal. The Body therefore is urg'd unto C by the Force Bg , which tends unto *Prop. 33.* S the Center. And so in all the Points, C, D, E, F . 1. *Q. E. D.*

Corollary (2.) "Seeing therefore in the Motion of the primary Planets, the Areas made by Rays, or right Lines drawn from them unto the Sun, are always proportional to the Times, as all Astronomers know, the Planets are urg'd by a perpetual Force, which tends to the Sun. And the same thing is equally true of the secondary Planets, with Respect to their primary ones.]

PROP. XLI, Theorem.

Fig. 65.

IF a Triangle (AFB) be in the same Parallels with a Parallelogram (AL) and have the same, or an equal Base (AB) it is half of the Parallelogram.

- (a) *Per* 37. Draw CB . The Triangles AFB , ACB are (a) equal.
 38. *l.* 1. But ACB is half of the Parallelogram AL (b.) There-
 (b) *Per* 34. fore AFB also is half of AL . *Q. E. D.*
l. 1.

Scholium.

Fig. 65.

FROM this Proposition, with the Scholium of *Prop.* 35. we learn, that the Area of whatsoever Triangle as AFB , is produced from half the Altitude FI multiplied into the Base AB , or half the Base multiplied into the Altitude. Wherefore one Side of a Triangle being known, and the Height, that is, the Perpendicular which falls upon the known Side from the opposite Angle, the Measure of the Triangle is given. As if the Base AB be of an 100 Feet, the Height FI , 85, multiply half the Base, 50 by 85, and you have the Area of the Triangle $AFB = 4250$ Feet Square. Further, the Altitude of a Triangle, when the Area of it is in all Points accessible, may be known mechanically as well as the Sides. But if the Area of it cannot be gone over, the Height may be found Geometrically by 12 and 13. *Lib.* 2. as we shall there shew.

In a Rectangled Triangle, the Height is the same with either of the Sides about the right Angle. Half of this therefore multiplied into the other Side adjacent to the right Angle, will give the Area of the Triangle.

PROP. XLII. Problem.

Fig. 66.

TO make a Parallelogram with an Angle equal to a given one (O), and equal to a given Triangle (ACB .)

(a) *Per*31. *l.* 1.

Bisect the Base AB in F . Through C draw CX parallel

(b) *Per*

(a) to AB . Make the Angle BAL equal to the given one

23. *l.* 1.

O (b.) Draw FI parallel (c) to AL . AL , FI shall be

(c) *Per*

that which was sought for.

1. *l.* 1.

For

For Let FC be drawn. The Parallelogram A I hath an Angle L A F equal to the given one O, and is equal to the given Triangle A C B; seeing that as well the Triangle A C B (d) as the Parallelogram A I (e) is double to the same Triangle A C F.

Corollary.

THE Triangle A C B being given, a Rectangle equal to it is had, if there be drawn a Line parallel to the Side A B, and A B being bisected in F, the Perpendicular B Q be erected. For the Rectangle under F B and Q B will be equal to the Triangle A C B.

P R O P. XLIII. Theorem.

IN a Parallelogram (as B L) the Complements (B O, O L) of those Parallelograms which are about the Diameter (R F, C S) are equal.

If through any Point of the Diameter A Q, as the Point O, C F be drawn parallel to the Side A B, and R S parallel to the Side B Q; the whole Parallelogram B L is divided into four Parallelograms, whereof two are about the Diameter R F, C S, the other two, B O, O L, are the Complements of these unto the whole Parallelogram B L. Their Equality is thus proved. The Triangles A B Q, A L Q are equal Likewise the Triangles A R O, O C Q (g) are equal to the Triangles A F O, O S Q (g) By the Therefore, if from the equals (b) A B Q, A L Q, you take away Equals, on this Side A R O, O C Q, on that A F O, O S Q; then B O and O L shall remain equal.

P R O P. XLIV. Problem.

UPON a given right Line (O S) to constitute a Parallelogram, in a given Angle (X, which Parallelogram shall be equal to a given Triangle (V.)

Make a Parallelogram R C equal to the given V. having its Angle R O C, equal to the given one X, and

- join the Side RO directly to the given Line OS , so as to make one right Line therewith. Then through S draw
 (b) *Per 31.* SQ (b) parallel to OC , which SQ let BC meet when it
l. 1. is produced unto Q . Then let a right Line, drawn through
 Q and O , meet BR produced unto A . Which done,
 through A draw AL , parallel to OS , which AL let CO
 and QS meet, when it is produced unto F and L ; the Par-
 allelogram OL is that which was required.
 (c) *By the* For OL (c) is equal to RC , that is, by the Construction,
foregoing. to the given Triangle V , and is at the given Line OS ; and
 (d) *Per 15.* (d) the Angle FOS is equal to the Angle ROC ; that is,
l. 1. by the Construction, equal to the given Angle X .

Scholium. " This Proposition contains a certain Geo-
 " metrical Division. For in the vulgar Arithmetical Di-
 " vision, the Number to be divided may justly be considered
Fig. 91. " as being a certain Rectangle. *e. g.* Let the Rectangle
 " AB comprehending 12 Square Feet, be to be divided by
 " 2; *i. e.* a Rectangle is to be found equal to that AB
 " of 12 Square Feet, one of whose Sides shall be only 2
 " Feet: From whence it comes to be enquired of what
 " Number the Side sought shall consist; which Side is to
 " be esteemed a certain Quotient of this Division. Which
 " thing is performed Geometrically after this manner.
 " With a pair of Compasses take the Line BD of 2 Feet,
 " and draw the Diagonal DEF . The Line AF is that
 " which is sought for. For the Complements EG and
 " EC are * equal; and in the Rectangle EG , one Side,
 * *Per 42.* " EH , is equal to the Line BD , which is of 2 Feet;
l. 1. " and the Side EI , is equal to AF .
 " This Kind of Division is called *Application*, because the
 " Rectangular Space AB is *applied* to the Line BD or
 " EH ; and hence it comes, that Division is often named
 " *Application*; respect being had to the Practice of the old
 " Geometricians, who always made more Use of Geome-
 " trical Construction, which requires only a Rule and a
 " Pair of compasses, than of Arithmetical Computation,
 " which is performed by Numbers.

PROP. XLV. Problem.

UPON a given Line ($I Q$) and in a given Fig. 69. Angle (H) to make a Parallelogram equal to a given Rectilinear Figure ($C B A$.)

Resolve the given Rectilinear into the Triangles A, B, C , by drawing the right Lines $F L, F I$.

Upon the given Line $I Q$ in the given Angle H , make (a) the Parallelogram $I V$ equal to the Triangle A . Then (a) *Per 44.* the right Line $I R$ being produced infinitely towards P ; *l. 1.* upon the right Line $R V$, in the Angle $V R P$, (b) make (b) *By the* the Parallelogram $R Z$ equal to the Triangle B . Again, *same.* upon the Line $S Z$, with the Angle $Z S P$, make the Parallelogram $S G$ equal to the Triangle C . This done, I say, $I G$ is the Parallelogram sought for.

For (c) the Angle $Z V R$ is equal to its Alternate $I R V$. (c) *Per 27.* But (d) $Q V R$ and $I R V$, are equal to two right Angles. *l. 1.* Therefore also $Q V R$ and $Z V R$, are equal to two right (d) *By the* ones. Therefore * $Q V$ and $Z V$ fall directly so as to make *same.* one Line right. After the same manner I might shew that * *Per 14.* $Q Z$ and $Z G$ make one right Line. Therefore the whole *l. 1.* $Q V Z G$ is one right Line, and is also parallel to $I X$, seeing by the Construction $Q V$ is parallel to $I P$. Now $X G$ also (e) is parallel to $I Q$. Seeing $X G$ is parallel to $S Z$, (e) *Per 30.* and $S Z$ to $R V$, and $R V$ to $I Q$. *l. 1.*

$I G$ therefore (f) is a Parallelogram; but that it is such (f) *Per* an one as was required, is manifest from the Construction. *Def. 35.*

[Corollary. " Hence is easily found the Excess whereby
" a greater Rectilinear Figure exceeds a lesser: To wit, if
" unto the same right Line $I Q$ be applied Parallelograms
" respectively equal to the two right-lined Figures. For
" that Parallelogram, by which the greater Rectilinear ex-
" ceeds the lesser, will give the Difference of them. *Q.E.I.*]

Scholium.

WE will here add a Problem that will be useful for the Practice of Proposition 14. *l. 2.*

A Quadrangular Figure, $B F$, being given, to describe Fig. 70. an equal Rectangle.

Resolve

Resolve it into Triangles by the right Line AC . From the opposite Angles, let down the Perpendiculars BO , FI . Bisect AC in S . From S erect the Perpendicular SL , equal to the two, BO , FI , put together. The Rectangle, comprehended under LS and SC , is equal to the given, BF . The Demonstration appears out of Proposition 41.

PROP. XLVI. Problem.

Fig. 71. FROM a given right Line (AB) to describe a Square.

- Erect two Perpendiculars equal to the given AB ; to wit, AC , BE , then join CE . I say, the Thing is done.
- (g) *By the Construction.* For seeing the two Angles A and B are (g) right ones, AC and BE shall (b) be parallel; but they are also (a) equal. Therefore CE and AB are (b) parallel and equal.
- (h) *Per 29. l. 1.* Therefore the Figure is Parallelogramical and Equilateral.
- (a) *By the Construction.* But all these Angles also are right ones (for seeing A and B are right Angles the opposite ones (c) E and C are right also) Therefore the Figure AE is a Square.
- (b) *Per 33. l. 1.* [" In the same manner you may easily describe a Rectangle, which hath the two unequal Sides given.]
- (c) *Per 34. l. 1.*

PROP. XLVII. Theorem.

Fig. 72. IN every Right-angled Triangle (as ABC) the Square of the Side (AC) which is opposite to the right Angle, is equal to the two Squares together of the two other Sides (AB , CB .)

- Let IC and BF be drawn; and BE parallel to AF . Now if to the right, and therefore equal Angles IAB , FAC , there be added the common Angle BAC , the Wholes, IAC and BAF shall be equal. But in the Triangles, IAC , BAF , the Sides which contain those equal DefSquare Angles, are equal (d) amongst themselves, to wit, IA , BA , AF , each to each. Therefore the Triangles IAC , FAB , (e) are equal. Which, because they stand upon

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en, B F.
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upon the same Bases, I A, F A, with the Parallelograms,
A B L I and Z A F E, and between the same Parallels, I A,
L B C, and A F, E Z B, they are halves (*f*) of those Pa-^(f)Per 41.
rallelograms. Therefore the Parallelograms, A B L I, *l.* 1.
Z A F E, as being Doubles of Equals, are equal betwixt
themselves. By the same reasoning, if right Lines, A X,
B R, were drawn, it might be shewn that the Parallelograms
E C, B X, are equal. Therefore the whole, A R is equal
to I B and B X, together. Q. E. D.

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It was taken for granted that LBC is parallel to IA , in order to which LB and BC must be one right Line. Now that they are so, is manifest from the 14th, seeing the Angles LBA and CBA , are both right ones by the Hypothesis.

Scholium.

THIS Theorem (which, *Prop. 31. l. 6, Euclid* extends unto all like or similar Figures) is commonly call'd the *Pythagoric* Theorem, from *Pythagoras*, the Inventor of it, who, as is attested by *Proclus*, *Vitruvius* and others, offer'd Sacrifices to the Muses, as supposing himself to have been helped by them in so excellent an Invention; in which thing he shew'd himself to be ignorant of God, the Lord of Sciences, the true and only Author of all Wisdom; or certainly, if he knew him, he glorified him not as God. There is frequent and notable Use of this Theorem through all the Mathematics; and in particular, it opens a Way unto the Knowledge of incommensurable Magnitudes, a main Secret of Geometrical Philosophy.

That the Side of a Square is incommensurable to the Diameter, is a Thing much celebrated amongst the old Philosophers, *Aristotle* and *Plato* especially; insomuch that *Plato* would say, that he who knows not this, is not a Man, but a Beast. Now the Knowledge of this Mystery seems to have taken its Rise out of this 47th Proposition For seeing in the Square A E, the Angle A is a right Angle, *Fig. 71.* the Square of the Diameter C B shall be equal to both the Squares of the Sides, A B, A C, and therefore double to one of them. Wherefore seeing the Square of C B is 2, and the Square of the Side A B is 1, or Unity, the Diameter C B shall be the Square Root of 2, and the Side A B the Square Root of Unity, it self; the *Ratio* of which Quantities

Quantities (as it will be demonstrated in its Place) cannot be explicated in Numbers, and therefore they are incommensurable.

And by this one Argument alone, if all others were wanting, it might evidently be made out, that Geometrical Magnitudes cannot be made up of a definite Number of Points: for otherwise none would be incommensurable; forasmuch as a Point would be the common Measure of all.

To these Things we will subjoin three Problems, which are deduced out of the present Proposition, and are of frequent Use.

Problem 1.

Fig. 73. **I**F any Number of Squares are given, to make one equal to them altogether.

Let there be three or more Squares given, whose Sides are $A B$, $B C$, $C E$. Make the right Angle $F B Z$, having indefinite Sides, and unto the Sides of it transfer $A B$ and $B C$, and then join $A C$. The Square of $A C$ shall be equal to

(a) *Per 47.* the Squares of $A B$ and $B C$ together (a.) Then transfer $A C$ from B unto X , and $C E$ the third given Side, transfer from B unto E , and join $F X$; the Square of $E X$ shall

(b) *By the same.* be equal (b) to the Squares of $E B$ (or $E C$) and $B X$ together; that is, equal to the three given Squares, whose Sides are $A B$, $B C$, $C E$: And so on as long as you please.

Problem 2.

Fig. 74. **T**WO unequal right Lines being given ($A B$, $B C$) to determine that Square, whereby the Square of the greater ($A B$) exceeds the Square of the less ($B C$).

From the Center B , with the Interval $A B$, describe a Circle. Then from C erect a Perpendicular $C E$, cutting the Circumference in E . The Square of $C E$ is the Excess or Difference which is sought for.

For let $E B$ be drawn. The Square of $B E$, that is, of $A B$ is equal to the Squares (a) of $B C$ and $C E$ together. (a) *Per 47.* Therefore, &c.

Problem 3.

Fig. 75. **A**NY two Sides of a Right angled Triangle being known, to find the Third.

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Let the Sides containing the right Angle be A B, A C, the one of 6 Feet the other of 8. You are to find of how many Feet the Side C B, which is opposite to the right Angle, is. To do which, multiply 6 and 8 each of them by it self. From which Multiplication there will arise for the Squares of those two Sides 36 and 64; the Sum of which is 100. The square Root of 100, which is 10, gives the Feet of the Side B C, whose Quantity was sought. This Demonstration offers it self in and from this 47th Proposition, for the Sum of the Squares B A and C A is equal to the Square of B C. Therefore the Root of the Sum of them is equal to the Root or Side B C.

Then let the Sides A B, B C be known, the one of 6 Feet, the other of 10, you are now to find A C. Take the Square of the Side A B which is 36, out of the Square of the Side B C = 100. The Remainder 64 shall be the Square of the Side A C. The Root therefore of 64, which is 8, gives the Feet of the Side A C.

Corollary. " From hence we derive the Original of *Fig. 92*.
 " the Tables of Sines, Tangents and Secants. For, In-
 " stance, let A C the Semi-diameter of the Circle be of
 " 100,000 Parts, and the Angle B A D of 30 Degrees.
 " Because the Chord or Subtense of 60 Degrees is * equal * *Per Corol.*
 " to A C the Semi-diameter; B D the Sine of 30 Degrees *1. Prop. 15.*
 " shall be equal to half the Semi-diameter, or $\frac{1}{2}$ A C; and *1. A. and Co-*
 " therefore shall contain 50,000 Parts. But now in the *rol. 2. Prop.*
 " right-angled Triangle A D B, the Square of A B is equal *3. 1. 3.*
 " to the Squares of A D and B D. Wherefore let the
 " Semi-diameter A B be squared (by multiplying 100,000
 " by 100,000) and from that Square subtract the Square of
 " B D; The Remainder shall be the Square of A D, or
 " of the Cosine equal to it B F; out of which extract the
 " square Root, and you will have the Line B F or A D.
 " Then by this following Analogy, A B : B D :: A E :
 " C E, or A D : B D :: A C : C E, will be had the Tan-
 " gent C E. And then lastly, if the Square of A C be
 " added to the Square of C E, the Root of the Sum being
 " extracted will be the Secant A E. *Q. E. I.*

PROP. XLVIII. Theorem.

I*F in a Triangle the Square of one of the Sides (A B) be equal to the two Squares of the other Sides (A C, B C) taken together, the Angle (A C B) which the two other Sides contain, is a right Angle.*

Fig. 76.

If not, the Angle A C B will be greater or less than a right Angle. In either of which Cases (as it will be demonstrated, *Prop. 12, 13, l. 2* which Propositions depend not on this) the Square of A B will not be equal to the Squares of A C, B C together; which is contrary to the Hypothesis.

- (a) *Per 47.* Or thus. Draw F C perpendicular to A C, and equal to
l. 1. C B, and join A F. The Square of A F is (a) equal to
 (b) *By the* the Squares of F C, C A together; that is, (b) to the
Construc- Squares of B C, C A; that is, by the Hypothesis, to the
tion. Square of A B. Therefore the right Lines A F, A B are
 (c) *Per 8.* equal. Because therefore the Triangles X and Z are mu-
l. 1. tually equilateral, the Angles at C (c) are equal. There-
 (d) *Per* fore they are both right Angles (d). *Q. E. D.*
Def. 14.



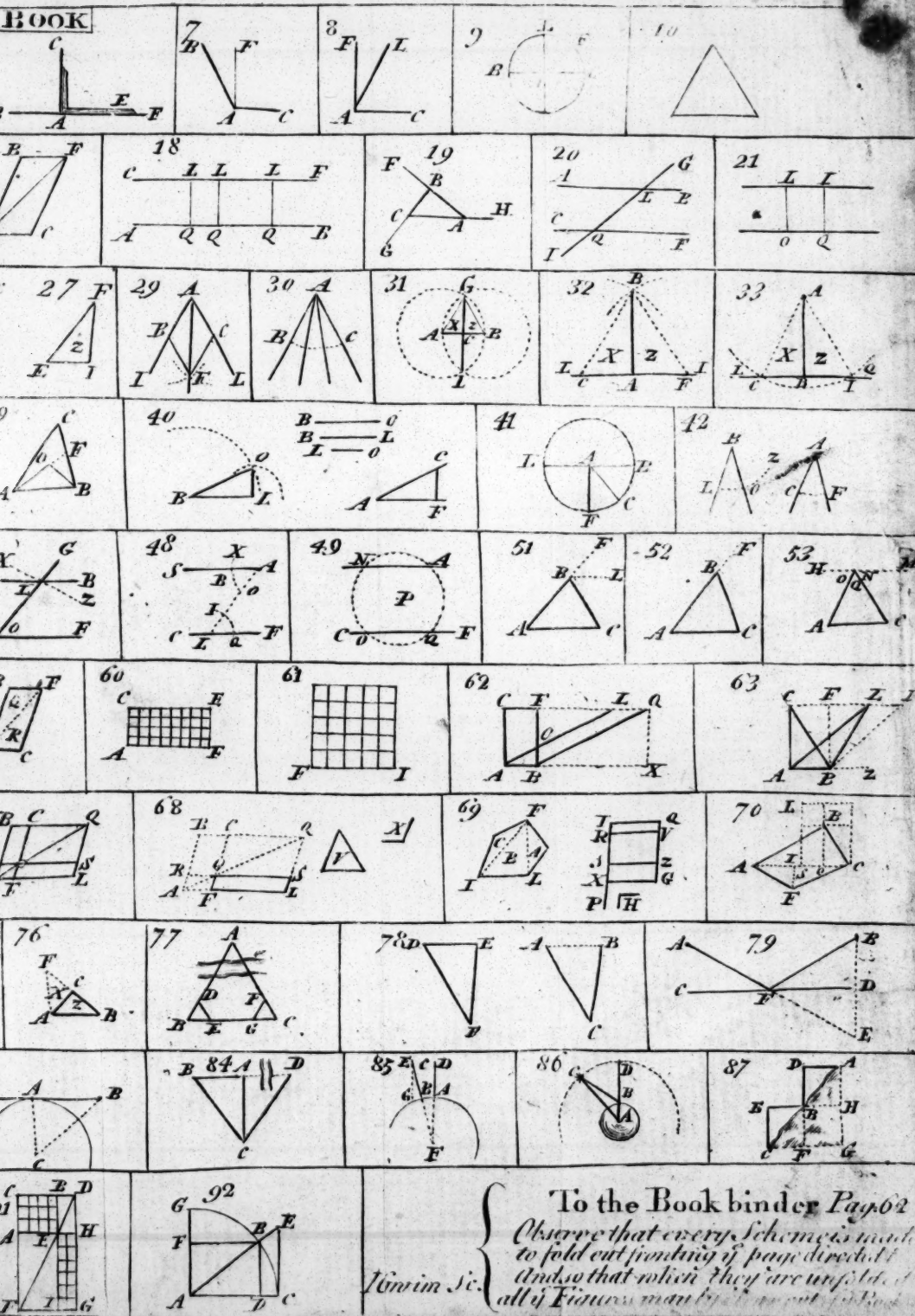
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B O O K II.

THIS Book is small in Bulk, but great indeed in the Excellence and Usefulness of its Theorems. Young Beginners will not, I know what I say, be at first able to discover it; but being further advanced, they will, from their own Experience, and with the greatest Certainty, apprehend that it is most true.

A DEFINITION.

A Right-angled Parallelogram (as A E) which is wont simply, and without any Addition, to be call'd a Rect-*Fig. 60.*
angle) is said to be contain'd under the two Lines (A C, A F) which determine the Magnitude of it. *l. 1.*

For the one of them A C determines the Height, the other A F the Breadth of it. Now, if the Side A C be understood to be carried perpendicularly along the whole A F, or A F along A C, by that Motion the Rectangle or its Area will be produced. Wherefore a Rectangle is rightly said to be produced from the drawing of two Lines into one another, or the Multiplication of them one by the other. When therefore you have these Words. [the Rectangle under (or of) A C, C B,] or for Brevity's Sake, [the Rectangle A C B] there is meant that Rectangle which is contained under A C and C B, multiply'd one into the other. In like manner, when we say the Rectangle under A B, B C, or the Rectangle A B C, there is design'd the Rectangle contained *Fig. 2.*
l. 2.

tain'd under the right Lines $A B$ and $B C$, multiply'd by one another.

Moreover, of Rectangles some are Oblong, some are Square. The Oblong Rectangle is that which hath its contiguous Sides unequal, or which is contained under two unequal right Lines. The Square Rectangle, that which is contain'd under two equal right Lines.

PROPOSITION I. Theorem.

Fig. 1. 1. 2. **I**f there be two right Lines ($A B$, $A C$), whereof $A B$ is divided into as many Parts as you will ($A E$, $E F$, $F C$;) the Rectangle compriz'd under those two ($A B$, $A C$) is equal to all the Rectangles together, which are contain'd under the undivided Line ($A B$) and the several Parts of the divided Line ($A E$, $E F$, $F C$.)

Make $A B$ perpendicular to $A C$, thro' B draw the infinite Line $B R$ parallel to $A C$. From E , F , C , erect the Perpendiculars $E I$, $F L$, $C Q$. $B C$ will be a Rectangle under $A B$ and $A C$; and is equal to the Rectangles $B E I F$, $L C$; that is, (because as well $I E$ as $L F$ are equal

* *Per 29,* * to $A B$) equal to the Rectangles under $A B$, $A E$; $A B$, $E F$; $A B$, $C F$.

Scholium.

THE ten first Theorems of this Book are true also of Numbers, if they as Lines be divided into Parts. The numerical Rectangles are produced from the Multiplication of two Numbers, and the numerical Squares from the Multiplication of the same Number by itself.

[" Let the undivided Number be 9, and the divided one 12. The Rectangle which is from 9 multiplying 12=108 will be equal to the three Rectangles, 27, 36, and 45, which are produced from 9 multiplied by 3, and 4 and 5, respectively and separately. Or let the Number 432 be as it were a Multiplicand divided into 400, and 30 and 2; and the Number 8 an undivided Multiplier; $8 \times 432 = 3456$ will be equal to $8 \times 400 = 3200$, $8 \times 30 = 240 + 8 \times 2 = 16$. And from this Proposition therefore the Demonstration of Multiplication is to be deriv'd.]

P R O

PROP. II. Theorem.

If the right Line (A B) be cut any where (as Fig. 2. in C,) the two Rectangles under the whole A B) and the Parts (A C, C B) are equal to the Square of the whole Line (A B.)

[" For A D is the Square of the whole, and A H, C D are Rectangles under the Whole A B, and the Parts A C, C B.] *Fig. 17.*

[" Let the Number 8 be divided into 5 and 3; the Square of the Whole $8 \times 8 = 64$, is equal to the Rectangles $8 \times 3 = 24$, $+ 8 \times 5 = 40$.]

PROP. III. Theorem.

ET a right Line, as (A B) be cut any where, Fig. 3. (as for instance in C,) the Rectangle contain'd under the Whole A B, and either of the parts, (B C) is equal to the Rectangle under the parts (A C, C B) together with the Square of the said Part (B C.)

[" For A F is the Rectangle under the whole Line A B, and the Part A C; and C F is the Rectangle under the Parts, as A E is the Square of the Part A C.] *Fig. 18.*

[" In Numbers. Let the Number 7 be divided into the Parts 3 and 4. The Rectangle of $7 \times 3 = 21$ is equal to the Rectangle of $3 \times 4 = 12$, together with the Square $3 \times 3 = 9$. In like manner $7 \times 4 = 28$, is equal to the Rectangle $3 \times 4 = 12$ + the Square $4 \times 4 = 16$.]

PROP. IV. Theorem.

ET a right Line, as (F L) be cut any where, Fig. 4. as in (O,) the Square of the Whole shall be equal to the Squares of the Parts (F O, O L) and to two Rectangles contain'd under the Parts (F O, O L.)

[" For F D is the Square of the Whole, and C G and C L the Square of the Parts; and C F, C D, two Rectangles under the Parts.] *Fig. 19.*

E

" In

" In Numbers. Let the Number 10 be divided into
 " two Parts, 7 and 3. The Square of $10 \times 10 = 100$
 " equal to the Squares of the Parts $7 \times 7 = 49$, and $3 \times 3 = 9$, and to the two Rectangles $7 \times 3 = 21$, and $7 \times 3 = 21$.
 " And on this Proposition depends the Extraction of the
 " Square Root.

Fig. 19.

Corollary (1.) " Hence it is manifest, that the Parallelograms about the Diameter of a Square, (O I, H K) are equal to the Squares.

(2.) " As likewise, that the Diameter of every Square bisects the Angles of it.

(3.) " And that the Square of half the Line is a fourth Part of the Square of the whole Line. For in that Case the Rectangles and Squares end in four equal Squares.

PROP. V. Theorem.

Fig. 5.

IF a right Line, as (Q X) be cut equally in (S) and unequally in (S), the Rectangle contained under the unequal Parts (Q S, S X) taken together with the Square of the intermediate Part (R S) shall be equal to the Square of the whole Line (Q R.)

Fig. 20.

[" For Q H is the Rectangle under the unequal Parts, and L G the Square of the intermediate Part, and the Square of half the Line; and therefore, because the Rectangle Q L is equal to the Rectangle S F, and the Square of the Space is common to both, the Proposition is manifest.]

" Let the Number 8 be divided equally, that is, into 4 and 4, and unequally into 5 and 3. The Rectangle of $5 \times 3 = 15$ together with the Square $1 \times 1 = 1$ shall be equal to the Square $4 \times 4 = 16$.]

PROP. VI. Theorem.

Fig. 6.

IF a right Line (A B) be divided into two equal Parts in C, and to it a certain right Line (B F) be adjoin'd; the Rectangle contained under the whole compound Line (A F) and the adjoin'd one (B F) taken together with the Square

half the Line (CB) shall be equal to the Square (CF) which is compounded of half the Line (CB) and the adjoin'd one.

[" For AN is the Rectangle under the whole compound Line and the adjoin'd one; and GK the Square of half the Line AB ; and CE the Square of the Line compounded of half the Line AB , and that which was added. Wherefore, because the Rectangle HE is equal to the Rectangle AK , and the rest of the Space is common to both, AN and KG is equal to CE . *Q. E. D.*]

[" If the Number 6 be divided into the two equal Parts, 3 and 3; and to it be added the Number 2; the Rectangle of $8 \times 2 = 16$, taken together with the Square $3 \times 3 = 9$, shall be equal to the Square $5 \times 5 = 25$.]

Corollary. " Hence, with *Maurolicus*, with one single Observation, we learn to measure the Diameter of the Earth. Let the Altitude of the Mountain AD be known, *Fig. 22.* and AB the Line touching the Earth be known by measuring. Let the Line DE be cut into two equal Parts in the Center C , and to it be added the Line AD . Now, because the Rectangle under AE , AD , together with the Square of DC , is by this Proposition equal to the Square of AC , that is, equal to the * Squares * *Per 17.* of the Lines AB , BC : From hence it follows, that if *l. 1.* you take away on both Sides the Square of CD or CB , the Rectangle which is under AE , AD is equal to the Square of AB . Therefore let the known Square of AB be divided by the known Altitude of the Mountain AD , and the Quotient will give the Line AE . From which subtract the known Altitude of the Mountain AD , the remaining Line DE will be the Diameter of the Earth. *Q. E. I.*

PROP. VII. Theorem.

If a right Line (AB) be cut any where, (as in *Fig. 7.* C), the Square of the whole Line (AB) taken together with the Square of either of the Segments (C) is equal to two Rectangles contained under the whole (AB) and that Segment (AC), together with the Square of the other Segment (CB .)

E 2

[" For

Fig. 23.

[" For EB is the Square of the whole Line, and A
 " the Square of the Part AC. But the two Rectang
 " under the whole Line, and that Part EI, HL, togeth
 " with GB, the Square of the other Part, possess t
 " same Space that EB and the Square of AC do
 " Therefore they are equal to EB and the Square of AC
 " Let the Number 13 be divided into any two Part
 " as 9 and 4. The Square $13 \times 13 = 169$, together w
 " that $9 \times 9 = 81$, is equal to $13 \times 9 = 117$, and $13 \times 9 = 117$
 " and the Square $4 \times 4 = 16$.]

PROP. VIII. Theorem.

Fig. 8.

IF a right Line (LF) be divided into two equal
 Parts in (I,) and to it a certain right Line
 adjoin'd (FO;) the Rectangle (L I O,) which
 contain'd under the half of the Line (L I) and
 the Line (I O) that is compounded of half
 aforesaid Line, and the Line adjoin'd, this Re
 angle taken four times, together with the Squ
 of the adjoin'd Line (FO,) shall be equal to
 Square of the whole compound Line (LO.)

Fig. 24.

[" For AL is the Square of the whole Compound,
 " taining four equal Rectangles under LI and IO (to
 " DR, BQ, RO, and the fourth made up of LR
 " QH added together) and with those four Rectangles
 " Square HE. From whence the Proportion is man
 " Let the Number 12 be divided into 6 and 6; and
 " Number 4 be added to it. The four Rectangles, 1
 " $= 240$ and $4 \times 4 = 16$ are equal to the Square $16 \times 16 =$

PROP. IX. Theorem.

Fig. 9.

IF a right Line (AC) be divided equally
 (B) and unequally in (F,) the Squares of
 unequal Parts (AF, FC) will be double to
 Squares of half the Line (AB,) and of the
 intermediate Part (BF.)

[" Let BE be equal and perpendicular to BA. From hence the Construction being made, as the Figure shews, the Lines AB, BE, BC will be equal: As also the Lines EG, GQ will be equal. The Angles AEC, ABE, CBE, EGQ, QFC will be right; and the Angles AEB, BEC, ECA, CQF, EQG half right ones. From whence the Square of AE will be double * to the Square of AB, which is half of AC, and the Square of EQ double to the Square of GQ or BF the intermediate Line. But the Squares of AE and EQ are † equal † to the Square of AQ, that is, to the Squares of AF and FQ or FC the unequal Parts. *Q E D.* By the same.

[" Let the Number 32 be divided equally into 16 and 16, and unequally into 20 and 12. The Square $20 \times 20 = 400$, with the Square $12 \times 12 = 144$, are double to the Squares of $16 \times 16 = 256$ and $4 \times 4 = 16$.]

PROP. X. Theorem.

If a right Line (FI) be divided into two equal Parts in (L, and to it a certain right Line (IO) be adjoin'd; the Square of the whole compound Line (FO,) taken together with the square of the additional Line (IO,) shall be double to the Squares, which are described upon the half Line (FL,) and (LO) that which is compounded of half the Line (FI) and the additional Line. Fig. 10.

[" For a Construction being supposed not unlike to the former; the Square of FE, is double to the Square of the half Line FL, and the Square of EG is double to the * Square of EQ or LO, which is compounded of the half Line and the additional one. But the Squares of FE and EG are equal to the Square FG; that is, to the Square of FO, the whole compound Line, taken together with the Square of OG or OI the additional Line. *Q. E. D.* Fig. 26.

[" Let the Number 40 be divided into 20 and 20, and to it let there be added the Number 14. The Square $54 \times 54 = 2916$, with the Square $14 \times 14 = 196$ are double to the Square of $20 \times 20 = 400$, taken together with $34 \times 34 = 1156$.]

PROP. XI. Problem.

Fig. 11. **S**O to cut the given right Line (AB) in (C) that the Rectangle (ABC) which is contained under the whole Line and one Part, shall be equal to the Square of the other Part (AC .)

From A erect a perpendicular AF equal to AB . Bisection of AF in X . Draw the right Line XB ; from the Line XB drawn forth, cut off XI equal to XB . Then cut off AI equal to AI . I say the Thing is done.

For let the Square $BAFS$ be perfected; and a Perpendicular being drawn through C , let the Rectangle $FILA$ be perfected also. Because FA is bisected in X , and to it is added AI ; there shall be

(a) Per 6.

l. 2.

(b) By the Construction.

(c) Per 47.

l. 1.

$$\left. \begin{array}{l} \text{The Rect. } FIA \\ + \\ \text{Square of } XA \end{array} \right\} = (a) \text{ to the Square of } XI$$

$$\left. \begin{array}{l} \text{That is, } = \text{to the Square of } XB \text{ (b)} \\ \text{That is, } = \text{to the Squares of } AB \text{ (c)} \\ \quad \quad \quad AX \end{array} \right\}$$

Therefore let there be taken away on both Sides the Square of XA ; there will remain

the Rectangle FIA or FL .

$= AS$ the Square of the Line BA ;

Wherefore again, the common Rectangle AO being taken away,

AL will remain equal to CS .

But AL is the Square of the Line AC , seeing by the Construction AC and AI are equal. And CS is the Rectangle ABC , forasmuch as BS is equal to AB . Therefore the Rectangle ABC is equal to the Square of AC . Therefore we have cut the Line AB , as it was required.

Scholium.

THE Ten first Propositions of this Book are true also of Numbers: But this Eleventh cannot be exemplified in Numbers; for no Number can be so divided that the Product of the whole multiplied by one Part shall be equal to the Square of the other. The Force of this Section of Line is wonderful. For which, see *Prop. 30. Lib. 6.*

P R O

PROP. XII. Theorem.

IN an Obtuse-angled Triangle (ACB), the Square of the Side (AB) opposite to the obtuse Angle (C), exceeds the Squares of the other Sides (AC, CB), by the Rectangle (BCF) twice taken; which same Rectangle is comprized under (BC) one of the Sides containing the obtuse Angle, and the Line (CF) which is intercepted betwixt the Perpendicular (AF) and the obtuse Angle.

Fig. 12.

The Square AB is equal to the Squares of AF } (a.) (a) Per 47.
 BF } l. 1.

But the Square of BF is equal to the Squares of FC , BC , with the Rectangle FCB twice taken (b.) Therefore (b) Per 4.
 you substitute these for the Square of BF ; then the l. 2.

square of AB is equal to AF Square }
 FC Square }
 CB Square }
 and Rectangle BCF twice

But the Squares of AF, FC are (c) equal to the Square of (c) Per 47.
 C . Wherefore this being substituted for them, l. 1.

AB Square is equal to AC Square }
 CB Square }
 + Rectangle BCF twice

PROP. XIII. Theorem.

IN any Triangle whatsoever (as ACB) the Square of the Side (AB) opposite to an acute Angle (C) is exceeded by the Squares of the other Sides (AC, CB) by the Rectangle (BCF) twice taken; which same Rectangle is contain'd under (BC) one of the Sides comprehending the acute Angle (C); and the Line (FC) which is intercepted betwixt the Perpendicular (AF)

Fig. 13, 14.

let

let fall upon the Side (BC) from its opposite Angle (A) and the acute Angle (C.)

(d) *Per* 4. The Square of BC is equal to (d) the Rectan. BFC twice.
l. 2.

(c) *Per* 47.
l. 1.

And AC Square is equal to (e) CF Square
+ FA Square

Wherefore the { BC Squ. } are equal to Rect. BFC
two together { AC Squ. }

+ BF Square
+ 2 FC Square
+ AF Square

But the Rectangle BFC twice, together with the Square of FC twice, is (a) equal to the Rectangle BCF twice.
(a) *Per* 3. Therefore this being substituted for them.
l. 2.

BC Squ. } are equal to the Rectang. BCF twice
+ AC Squ. }

(b) *Per* 47. But the Squares of AF, BF are equal to (b) the Square of AB. Therefore this being substituted for them.
l. 1.

BC Squ. } are equal to the Rectangle BCF twice
+ AC Squ. }

That is, BC Square + AC Square do exceed AB Square by the Rectangle BCF twice taken.

Corollary.

Fig. 15. THE Proposition is true, although the Perpendicular falleth without the Triangle. And the Demonstration is almost the same.

(c) *Per* 12. [“ More briefly thus. AC q = (c) AB q + CBQ
l. 2. “ 2 CBF. And on both Sides CBq, then AC + CB

(d) *Per* 3. “ = AB q + 2 CBq + 2 CBF = (d) AB q + 2 BC
l. 2. “ Q. E. D.

Scholium.

FROM this Proposition, and the 47th of the former Book, we have the Measure of any Triangle whatsoever, whose three Sides are known, although the Area be altogether inaccessible. For by the help of these Theorems, the Perpendicular is known, albeit the Impediments of the Place should not permit us to mark it out. But Note, That the Perpendicular, multiplied by half the Side on which it falls, produceth the Area of the Triangle, as appears out of the *Scholium* of the 41st Proposition, *Lib. 1.*

Let there be any Triangle (as A B C) having its Sides known. It is required to give the Perpendicular A F, which falls from the given Angle A upon the opposite Side C B. *Fig. 15, or 14.*

Take the Square of the Side A B opposite to the acute Angle C, out of the Sum of the Squares of A C, and B C. By the 13th, the Remainder shall be the Rectangle B C F twice taken. Divide half of the Remainder, that is, the Rectangle B C F by the known Side B C; thence will arise the right Line C F. Take the Square of the right Line C F out of the Square of A C. The Remainder will give (a) the Square of A F, whose square Root will give the Perpendicular A F. *(a) Per Prob 2. Schol post*

This thing also may be obtained out of the 12th Proposition. But the 13th sufficeth, forasmuch as in every Triangle the Perpendicular let fall from some one of the Angles into the opposite Side, falls within the Triangles. *47. lib. 1.*

PROP. XIV. Problem.

THE Right-lin'd Figure (Q X Z) being given, to make a Square equal to it. *Fig. 68.*

Make (b) a Rectangular Parallelogram C I equal to the Rectilinear Q X Z; the Sides of which Parallelogram, if they shall be equal, you have already made the Square which was required; if they be unequal draw forth the greater Side I A unto L, until A L shall be equal to A C. Then bisect I L in Z; from which, as from a Center through *(b) Per 45. 1.*

through I and L describe a Circle, and let C A be produced till it meets the Circumference in B. The Square of the right Line A B is equal to the given Rectangle Q X Z.

(c) *Per* 5. For let the right Line Z B be drawn; because I L is cut
l. 2. equally in Z, and unequally in A; the Rectangle

(d) *By the* I A L }
Construc- + Z A Square } are equal (c) to Z L Square, that is, equal
tion. to (d) Z B Square, that is, equal to (e) Z A Square + A B

(e) *Per* 47. Square.

l. 1. Taking away therefore on both sides the common Z A q,
there remains

Rect. I A L equal to A B q; that is,

Because A C and A L are equal, the Rect. C I equal to
A B Square, and consequently A B Square equal the Recti-

(g) *By the* linear (g) Q X Z.

Construc-
tion.

Scholium.

EUCLID's Construction of this Problem requires that the
given Rectilinear reduced unto a Rectangle by *Prop.* 45.

l. 1. Which Reduction being operose enough, the Problem
perhaps will more readily be dispatch'd after this manner.

Let the given Rectilinear be resolv'd into as many Qua-

(a) *Per* drangles (X, Z) as it can. Then to each Quadrangle (a)
Schol. P. 45 make an equal Rectangle. If there remain, as here it hap-

l. 1. pens, one Triangle (Q.) to it also (b) make a Rectangle

(b) *Per* equal. Then to each Rectangle by this 14th, l. 2. make an
Corol. p. 42. equal Square; and lastly, to all these Squares let one equal

l. 1. one be made (c) This will be equal to the given Rectili-

(c) *Per* near Q X Z.

Prob. 1.

Schol. p. 47.

l. 1.



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Elements of EUCLID.

B O O K III.

THE Fundamental Properties of the most perfect amongst Plain Figures are demonstrated in this Book. The Usefulness of the Book is manifest by this one Thing alone, that it treats of a Circle, that abundant Source of admirable Things through the whole Mathematicks. The more famous Theorems are 16, 20, 21, 22, 31, 32, 35, 36.

D E F I N I T I O N S.

THOSE Circles are equal, whose Diameters or Semi-diameters are equal. *Fig. 20 l. 3.*

2. A right Line (F B) is said to touch a Circle, when it doth so meet it in the Point (B) that albeit it be produced, it doth not cut it.

3. Circles are said to touch one another, when they do meet that they do not cut each other. *Fig. 13, 14*

4. In a Circle the right Lines (B C, F L) are said to be equi-distant from the Center (A,) when the Perpendiculars which are let fall upon them from the Center (A O, A I) are equal. *Fig. 18.*

5. Segments or Portions of a Circle are the Parts into which the right Line (C E) which cuts the Circle doth divide it. *Fig. 37.*

6. An Angle in a Segment is that (B Q C) which is contained under the right Lines, which are drawn unto one Point of the Circumference (Q) from the Ends of the Segment (B C) *Fig. 33.*

7. The

Fig. 33. 7. The Angle (C Q B) is said to stand upon the Circumference (B O C) as being opposite to it.

Fig. 11. 8. A Sector is that Part of a Circle which is contain'd by two Semi diameters, as (A B, A F) and an Arch as (B F or B C F) intercepted betwixt the Semi-diameters.

PROPOSITION I. Problem.

Fig. 1. 1. 3. *To find the Center of a given Circle.*

Let the right Line (B C) be drawn in the Circle at random, which biseft in Q. Through Q draw the Perpendicular L F, which biseft in A. A shall be the Center.

If you deny it; let the Center be O, which is without the right Line F L (for in F L it cannot be, forasmuch as this Line is divided every where unequal'y but in A) and let there be drawn B O, Q O, C O. Because therefore you suppose O to be the Center, B O, C O must be equal; and the Triangles B O Q, C O Q must be equilateral to each other; seeing by the Construction B Q and C Q are equal, and Q O is common. Therefore the Angle O Q C (a) is equal to the Angle O Q B. Therefore O Q C is a right Angle (b) and consequently equal to L Q C, which is a right one by Construction, a Part to the Whole. Which is absurd.

(a) *Per 8.*

l. 1.

(b) *Per*

Def. 14.

l. 1.

Corollary.

FROM what hath been demonstrated it appears, that the right Line (L F) cuts another right Line B C into two equal Parts and perpendicularly, the Center is in the Line that cuts the other.

Fig. 2.

The Center of a Circle is very easily found by a Square: the top of it (Q) being applied to the Circumference; for if the right Line D E, joining the Points D and E, in which the Sides of the Square cut the Circumference, be bisefted in A, (A) shall be the Center. The Demonstration whereof depends on the 31st Proposition, *Lib. 3.*

PROP. II. Theorem.

I*F in the Circumference of a Circle there be taken two Points (C and B) the right Line which is drawn through them falls entirely within the Circle.*

Let there be taken in the Line BC any Point whatsoever, Fig. 2. as O, and from the Center A, be drawn AB, AO, AC. Because AB, AC are equal, the Angles also B and C are (c) equal. Because therefore AOC is (d) greater than the (c) Per 5. internal one B, it shall be greater also than C. In the Tri- 1. angle therefore OAC, the Side AC subtending the greater (d) Per. Angle AOC, is (e) greater than the Side AO, subtending Corol. 1. the lesser Angle C. Seeing therefore AC reaches no far- Prop. 32. ther than from the Center to the Circumference, AO 1. shall not reach so far. Therefore the Point O shall fall (e) Per 19. within the Circle. The same thing may be shewed of any 1. other Point of the Line BC. Therefore BC falls wholly within the Circle.

The Proposition is also manifest from the very Notion of a right Line and a Circle.

Coroll. "Hence it follows, that a right Line touching a Circle, toucheth it in one single Point only. For if it toucheth the Circumference in two Points, it would be a right Line drawn thro two Points of the Circle, and consequently would fall within the Circle, contrary to the Definition of a Tangent. And by the like reasoning (in passing from Planes to Solids) it might be prov'd, that every Plane toucheth a Sphere only in one Point.

PROP. III. Theorem.

I*F in a Circle a right Line (BL) drawn thro' Fig. 3. the Center bisects another (CF) not drawn through the Center, it will cut it perpendicularly. And if it cut it perpendicularly, it will bisect it.*

Part I. From the Center (A) let there be drawn AC, AF. The Triangles X and Z are Equilateral to each other.

other. For CO , FO are by the Hypothesis equal, and AC , AF are so, because drawn from the Center; while AO is common to both. Therefore the Angles AOO , AOO are equal.

(a) *Per* 8. AOF are (a) equal. Therefore right (b) ones. Which was the first Part.

(b) *Per* Part II. Because by the Hypothesis AOC , AOF are equal Angles; AC Square shall (c) be equal to the Squares

Def. 14. of AO , OC together; and AF Square equal to the Squares

(c) *Per* 47. of AO , OF together. Seeing therefore the Squares of AC , AF are equal, the Squares of AO , OC together, will be

l. 1. also equal to the Squares of AO , OF together: Wherefore taking away the common Square AO , the Squares of OC , OF remain equal. And therefore the right Lines OC , OF are equal. Which was the other Part.

Corollary (1.) "Hence in every equilateral Triangle, and in that also which is only an *Isoceles*, a Line which falling from the top of the Angle, bisects the Base, is perpendicular to it. And on the contrary, a Line which falling from the top of the Angle is perpendicular to the Base, doth bisect it.

(2. "Hence it follows, that half of the Chord of every Arch, is the right Sine of half the Arch.

PROP. IV. Theorem.

Fig. 4. 5. **I***F in a Circle two right Lines, (BC , FL) not drawn both of them through the Center, cut each other, they cannot bisect each the other.*

Fig. 5. For if one of them LF passeth through the Center, it is manifest that it shall not be bisected by BC which doth not pass through the Center.

Fig. 4. If neither of them passes through the Center, from the Center A draw AO . If now BC , FL were both bisected,

(a) *By the* in O , the Angles AOC , AOL would (a) be right Angles, and consequently equal; the Whole to a Part, which is absurd.

PROP. V, VI. Theorems.

Fig. 6, 7. **C***ircles cutting each other, or inwardly touching one the other, have not the same Center.*

For

For if it were otherwise, the right Lines AB , ACF , drawn from the common Center A , would be equal; and AC would be equal to AF ; a Part to the Whole, because they are both equal to AB . Which is absurd.

PROP. VII. Theorem.

If in a Circle there be taken any Point besides Fig. 8. the Center (A), as the Point (C), and divers right Lines fall from thence unto the Circumference (as CB , CL , CO , CF);

1. (CB) which passeth through the Center, will be the greatest.

2. The remaining Part of the Diameter (CF) will be the least.

3. Of the rest that will be the greater, which is nearer to the greatest.

4. And no more than two equal Lines can be drawn from the said Point (C), which is different from the Center, unto the Circumference.

Part I. Let AL be drawn from the Center A . Because AL , AB are equal, the common Line AC being added to each. AC and AL together are equal to CB . But AL + AC are greater than LC (*b*). Therefore CB is greater than LC . In the same manner BC will be shew'd to be greater than any other. (*b*) *Per 20.*

Part II. From the Center A draw AO . AO (that is, AF) is less than AC , CO (*c*). Therefore taking away the common Line AC , CO remains greater than CF . In the same manner CF is prov'd to be less than CQ , or any other. (*c*) *By the same.*

Part III. In the Triangles COA , CLA , the Sides AO , AC , are equal to AO , AC , each to each. But the Angle LAC is greater than the Angle OAC . Therefore (*d*) the Base LC is greater than the Base OC . (*d*) *Per 24.*

Part IV. This is manifest from what goes before. For if there could be three drawn equal, CO , CL , CQ , there would be two on the same Side equal: Which is contrary to Part III. (*1.*)

Corollary

Corollary. " By the foregoing reasoning *Theodosius* gathered, that of the Arches of great Circles drawn upon the Surface of a Sphere, from any Point diverse from the Pole of a certain Circle unto that Circle, the greatest is that which passeth thro' the Pole of that Circle; the least, that which is drawn unto the opposite Point; and of the rest, that is the greater, which is nearest to the greatest; as also that no more than two equal Arches can be drawn from that Point unto the Circle. And in the like manner may the reader reason of himself on some other of the Propositions of this Book; it being very easy to pass from Planes to Solids in these Argumentations.

PROP. VIII. Theorem.

Fig. 9, 10. **I** F from a Point (*A*) taken without a Circle there be drawn unto the Circle the right Lines (*AB, AC, AF,*) or (*AO, AQ, AR;*)

1. Of those which fall upon the Concave Circumference, the greatest is (*AB*) which passeth through the Center (*Z*.)

2. Of the rest that is the greater, which is nearer to the greatest (*AB*.)

3. Of those which fall without the Circle, upon the convex Periphery, the least is (*AO*) which being produced would pass through the Center (*Z*.)

4. Of the rest, that which is nearer to the least, is less than that which is farther off.

5. No more than two equal Lines can be drawn unto the Circumference from the same Point (*A*), whether they fall within the Circle or only without.

Fig. 9.

Part I. From Center *Z* draw *ZC*; because *ZC, ZA* are equal, the common *AZ* being added to each $AZ + ZC$ are equal to *AB*. But $AZ + ZC$ are (*a*) greater than *AC*.

(a) Per 20.

Therefore *AB* is greater than *AC*. In like manner *AB* will be shewed to be greater than any other whatsoever.

l. 1.

Part II Draw *ZF*. Because in the Triangles *AZO, AZF*, the Sides *AZ, ZC* are equal to *AZ, ZF*, each

each; but the Angle AZC is greater than AZF , therefore the Base AC (*b*) will be greater than the Base AF . (*b*) *Per 24.*

Part III. Draw ZQ . The two Lines AQ, QZ are *l. 1.* greater than AZ (*c.*) Taking away therefore the Equals *Fig. 10.* Q, ZO , there remains AQ greater than AO . In the (*c*) *Per 20.* the same manner AO is prov'd less than any other. *l. 1.*

Part IV. Draw ZR . The right Lines AQ, QZ are less than AR, RZ (*d*); therefore the Equals ZQ, ZR (*d*) *Per 21.* being taken away, AR remains greater than AQ . *l. 1.*

Part V. This is manifest from the Four foregoing.

PROP. IX. Theorem.

From some Point within a Circle (as A) more than two equal right Lines can be drawn unto the Circumference; that Point is the Center. *Fig. 11.*

This is manifest from Part IV. of the 7th Proposition.

PROP. X. Theorem.

Circles cut each other in two Points only. *Fig. 12.*

For let them cut, if it may be, in more (B, C, F) from the Center of the Circle LQ , let there be drawn the Points BCF , the Lines AB, AC, AF ; these shall be equal. Because therefore from the Point A , within the Circle OS , there are drawn three equal Lines, AB, AC, AF , unto its Circumference, A must also be the Center (*a*) of the Circle OS . Therefore the Circles LQ , (*a*) *By the foregoing.* which cut one another, have the same Center. which contradicts the 5th Proposition.

PROP. XI. Theorem.

If two Circles touch each other inwardly, a right Line drawn through their Centers (A and I) shall pass through the Point of Contact (B.) *Fig. 13.*

If you deny it, let the Centers have, if it may be, that position, that a right Line passing through them, shall fall

fall without the Contact B, cutting the Circles in O and
Let the Center be A and C; and join A B, C B. Because
therefore C B, C O are equal, the common A C being
ded to each of them, $A C + C B$ shall be equal to A

(b) *Per 20.* But A C, C B are (b) greater than A B, that is, than

l. 1. (c). Therefore also A O is greater than A L, a Part

(c) *By the* the Whole. Which is absurd.

Definition
of a Circle.

PROP. XII. Theorem.

Fig. 14. **I** *F* Circles touch one another on the outside,
right Line, which joins the Centers, must pass
through the Point of Contact.

If it be denied, let the Centers be so placed, as for
stance in A and B, that the Line passing through them
not pass through the Contact S, but cut the Circles in O
Q. Let the Points A, S and B, S be joined. Then

(d) *Per 20.* B S together will (d) be greater than A B. But A S is

l. 1. equal to A O, and B S equal to B Q. Therefore A O

(c) *By the* B Q together will be greater than A B, a Part than
Definition Whole. Which cannot be.

of a Circle. [Corollary. " A right Line drawn from the Center
" one of the Circles through the Point of Contact,
" pass through the Center of the other.]

PROP. XIII. Theorem.

Fig. 15, 16. **C**ircles touch both one another, and a right
Line in a Point only.

Fig. 15. For let two Circles touch one another inwardly in a
of the Circumference L C, if it may be: Then a right

(f) *Per 11.* drawn through the Centers A and B will (f) pass thro
l. 3. the Point of Contact, as in C. Let there be drawn

A L, B L. Because therefore B L, B C are equal, (for
are drawn from the Center B unto the Circumference O
the common Line A B being added, A B, B L shall be e
to A C. But A C is equal to A L, for they are both d
from the Center A unto the Circumference L Q C. Th
fore A B, B L are equal to A L, contrary to *Proposition*
L. 1.

Then let the two Circles touch one another on the out-*Fig. 15.*
 in the Arch O L, if it may be. The right Line A P,
 ing the Centers, will pass through the Point of Contact
 as in O, for instance: Let A L, P L, be drawn. The *(a) Per 12.*
 Sides of the Triangle A L, P L, will be equal to A O, *l. 3.*

or the whole A P; contrary to *Proposition 20. L. 1.*
 astly, Let the right Line B F, and the Circle touch each
 er, if it may be, in some Part (C E :) Let there be
 wn unto the Center the right Lines C A, E A. The
 es C A, E A will then be equal: And therefore the Tri-
 e C A E is an *Isoceles*. Wherefore the Angles C and E
 are acute. And therefore a Perpendicular let fall unto *(b) Per*
 from the Center A, will fall betwixt E and C, *(c) as, Corol. 11.*
 nstance, in D. There will therefore both A C and A E *Prop. 32.*
 equal to the Perpendicular A D, which is absurd, and *l. 1.*

ary to *Corollary 14. p. 32.* and to *Proposition 47. L. 1. (c) Per*
Corol. 3.
Prop. 32.
l. 1.

Corollary.

Circles, whose Centers are in the same right Line, and *Fig. 17.*
 which cut it in the same Point B, do touch one an-
 in that Point only.

This Proposition is manifest from the very Notion of the
 s which are compared together. For neither can a
 Line and the curve Circumference of a Circle, or the
 s Curvatures of unequal Circumferences, or two Cur-
 res both convex, agree as to any Part of themselves.
 they would agree if they touched one another in some
 e and proper Part.

PROP. XIV. Theorem.

a Circle, equal right Lines (B C, F L) are *Fig. 18.*
 equally distant from the Center (A.) And
 Lines are equi-distant from the Center are

om the Center (A) let there be drawn (A C, A F.)
 ise A O, A I) at right Angles to B C, F L. Thus
 F L shall be bisected *(d) in O and I. (d) Per 3.*
 ing therefore the whole Lines B C, F L are sup-*l. 3.*
 equal, the halves also O C, I F must be equal, and
 quently the Squares of them are also equal. Seeing

F 2

therefore

- therefore the Squares of AC , AF are equal, and the Square of AC is equal to OC^2 and OA^2 , as also the Square of AF is equal to IF^2 and IA^2 (a): It follows, that
- (a) *Per* 47. AF is equal to IF , and IA (a): It follows, that
- l. 1.* two Squares OC^2 , OA^2 are equal to the two Squares IF^2 , IA^2 . Wherefore taking away the Squares OC^2 , IF^2 (which before were shewed to be equal) the Square of AO remains equal to the Square of AI . Therefore
- (b) *Per* Perpendiculars OA , AI are equal. Therefore (i) B
- Def. 4. l. 3.* FL are equi-distant from the Center. Which was the Part. Then for the converse of it;
- If the Distances AO , AI are supposed equal, then Squares of the equal right Lines being taken away, by the same Ratiocination it will be shewed, that the remaining Squares OC^2 , IF^2 are equal, and consequently the right Lines OC , IF are equal, which seeing they
- **Per* 3. *l. 3.* * halves of the right Lines BC , FL , these also must be equal. Which was the second Part.

PROP. XV. Theorem.

Fig. 19. **O***F right Lines described in a Circle, the greatest is the Diameter; and of the rest, that is the greatest, which is the nearest to the Center.*

- Let there be any Line, as RS different from the Diameter FL . From the Center draw AR , AS . The
- (c) *Per* 20. AR , AS , which are equal to the Diameter, are (c) greater
- l. 1.* than RS . Therefore, &c.

- Then let BI be nearer to the Center than XZ . From the Center unto them draw the Perpendiculars AC , AQ .
- (d) *Per* AQ shall be greater (d) than AC . Take therefore AO equal
- Def. 4. l. 3.* to AC , and through O draw RS perpendicular to AC , by which (e) will be equal to BI ; and let AR , AS , as in the
- (e) *By the foregoing.* AZ be join'd. Because therefore A is the Center, the Lines AR , AS shall be equal to AX , AZ . But the Angle RAX is greater than the Angle XAZ . Therefore the Base RZ is greater than the Base XZ (f). Q. E. D.
- (f) *Per* 24. that is BI , is greater than the Base XZ (f). Q. E. D.
- l. 1.*

PROP. XVI. Theorem.

Fig. 20. **A** *Right Line (IF) which being drawn through the Point (B), the Extremity of the Diameter (CB) is perpendicular thereto, falleth*

b. III.

it without the Circle, and toucheth it in (B).
neither can any right Line be drawn betwixt it
f and the Circle unto the Point of Contact (B),
t it shall cut the Circle.

Part I. Let there be taken in the Line I B F any Point
unto which, from the Center A, draw the Line A L.
cause, in the Triangle A B L, the Angle A B L is a right
e. by the Hypothesis, A L B shall be acute (g). There-(g) *Per*
e A L, which is opposite to the greater Angle B, will *Corol. 5. p.*
greater than A B, which is opposite to the lesser Angle *32. l. 1.*
(b). But A B reacheth only to the Circumference. (h) *Per*
therefore A L shall reach beyond the Circumference; and *19. l. 1.*
sequently fall without the Circle. Which was the first

Part II. Below B F, if it may be, let R B fall wholly
hout the Circle. Because F B A is a right Angle by the
ypothesis, R B A will be acute, and therefore A B is not
pendicular to B R. Therefore let there be drawn from
Center A to B R, the Perpendicular A O, which (a) (a) *Per*
fall towards R, and cut the Circle in Q. Therefore *Corol. 3.*
B, which is opposite to the greater Angle A O B, is *Prop. 32.*
ater than A O, which is opposite to the lesser, to wit, *l. 1.*
acute Angle O B A. But A B is equal to A Q:
therefore A Q also is greater than A O, a Part than the
ole.

Corollary.

HENCE it appears again, that the Contact of a right *Fig. 20.*
Line and a circular one, is only in one Point.
If from Centers taken in the same right Line infinitely *Fig. 17.*
tracted, there be described through B infinite Circles, as
ll lesser than the first B S C, as greater; they shall all
ch the right Line I F in the same one Point B.
Circles therefore growing into an Amplitude greater
any given one, approach always, even unto Infinity,
ter and nearer to the Tangent, but are never join'd to
otherwise than in one single Point of Contact; which
g, although it be most evident, is yet truly admirable.

Fig. 17. 4. From these Things it is manifest, that every Geometrical Line whatsoever is infinitely divisible. For there be drawn from some Point of the Diameter unto the Tangent the right Line A Q. Infinite Circles having Centers in the right Line B A, infinitely produced, touch the right Line I F by *Corollary 2.* of this, and one another by *Corollary*, p. 13. in one and the same Point B, and consequently are no where joined, either amongst themselves, with the right Line I F, but in the Point B only. Therefore it is necessary that they divide the right Line A Q into infinite Parts, that is, into Parts exceeding any Number assignable.

Fig. 20. 3. The Angle of Contingence or Contact L B Q, (that is to wit, which is contained under the Tangent and the Circumference) cannot be divided by any right Line.

Fig. 17. 6. Nevertheless, by Circumferences touching the Line I F in the same Point, it may be divided and diminished infinitely. And in this, and the third *Corollary*, lies the whole Mystery of Asymptotes, that is, of a right Line approaching unto an Hyperbola, together with itself infinitely produced, unto a Distance less than any given one, yet never concurring with it.

P R O P. XVII. Problem.

Fig. 26. FROM the given Point (B,) to draw a right Line, which shall touch a given Circle (O Q).

From A the Center of the given Circle, let there be drawn into the Point B, the right Line A B, cutting the Periphery in O. From the Center A describe through another Circle B C, and from O draw O C perpendicular to A B, which may meet the other Circle in C. Draw C A meeting the Circle O Q in I. The right Line drawn from B unto I, will touch the Circle O Q.

For because the Sides B A, I A, are equal to the Sides C A, O A, and the Angle A contain'd betwixt the equal Sides is common to both. In the Triangles I A B, O A C

(a) *Per 4.* the Angles A O C, A I B are also (a) equal. Therefore

l. 1. A I B is a right Angle. For A O C is a right one by

(b) *Per 16.* Construction. Therefore B I (b) toucheth the Circle in

l. 3.

Scholium

Scholium.

BY the 31st following, from the given Point O, a Line touching a given Circle (B Q) may be well drawn

us :
Let the right Line, joining the given Point O, and the Center A, be bisected in P. Then from the Center P, through A and O, describe a Circle, meeting the given Circle in B. The right Line O B will touch the Circle. For A B being join'd, the Angle A B O in the Semi-circle is a right one by *Prop. 31*. Therefore by *Prop. 16*. B toucheth the Circle B Q.

PROP. XVIII. Theorem.

If a right Line (C L) touch a Circle, a right Line (A B) drawn from the Center (A) unto the Point of Contact (B) is perpendicular to the tangent.

If it be denied, let some other right Line (as A F) be the perpendicular from the Center A. This will cut the Circle in O. Because therefore the Angle A F B is supposed to be a right one, A B F (c) must be acute. Therefore A B (c) *Per.* that is, A O) is greater than A F (d) ; a Part than the whole, which is absurd.

Coro! 5.
p. 32. l. 1.
(d) Per 19.
l. 1.

PROP. XIX. Theorem.

If a Line (B C) touch the Circle, and from the Point of Contact (A) there be rais'd A I) perpendicular to the Tangent, the Center will be in that Perpendicular.

If you deny it, let the Center be without A I in Z ; and let there be drawn unto the Contact the Line Z A. The Angle Z A C will be a right one (e) and therefore (e) equal to the Angle I A C, which, by the Hypothesis, is a right one ; that is, the Part will be equal to the Whole. which is absurd.

PROP.

PROP. XX. Theorem.

Fig. 30, 31, 32. **T**HE Angle at the Center (BAC) is double to the Angle (BFC) which is at the Circumference, when the same Arch (BC) is the Base of the Angles.

Fig. 30. Here are three Cases. In the first Case the Sides BA BF coincide. And then because AF , AC drawn from the Center are equal, there will be in the Triangle Z , the
(a) Per 5. l. 1. Angles F and C equal (*a*). But BAC is equal to the two
(b) Per 32. l. 1. Angles F and C (*b*). Therefore BAC is double of F .
Fig. 31. In the second Case, BA , CA fall within BF , CF , and then FAX being drawn, XAB , by the first Case, is double of XFB ; and XAC double of XFC . Therefore the whole BAC is double of the whole BFC .

Fig. 32. In the third Case, BF cuts AC , and the Angle BAC is without the Triangle BFC . Here let FAL be drawn. By the first Case, the whole LAC is double of the whole $LF C$, and LAB taken away, is double of $LF B$ taken away. Therefore the remaining Angle, BAC , is also double of the remaining one, BFC . *Q. E. D.*

Fig. 53. Corollary. "Hence we gather, that the Sides of every Triangle are to each other as the Sines of the Angles opposite to those Sides respectively. Let EFG be any Triangle; about which let a Circle be understood to be circumscrib'd (*c*) and from the Center of the Circle, let there be let down the Perpendiculars AB , AC , AD , which will * bisect the Subtenses. Now as EF is to EG so half EF (that is, EB) to half EG (that is, ED). But EB is the Sine of the Angle $\dagger BAE$, that is, half the Angle EAF , that is, of the whole Angle EFG , * opposite to the Sides EF ; and ED is the Sine of the Angle EAD , that is, of half the Angle EAF , that is, of the whole Angle EFG , which is opposite to the Side EG . Therefore EF is to EG , as the Sine of the Angle EFG , is to the Sine of the Angle EFG . *Q. E. D.* And from this one Proposition a great Part of Trigonometry is deduced. Which thing will be worth our Observation.

Corolla

Corollary (2.) " From the former Corollary we learn *Fig. 86.*
 " to measure the Distance of the Moon. For Astronomi-*l. 1.*
 " cal Observations giving us the Angle of the Diurnal Pa-
 " rallax * $B C A$, we find out the Distance of the Moon by * *Corol. 16.*
 " the following Proportion. As the Sine of the Angle *p. 32. l. 1.*
 " $A C B$, is to the Sine of the Angle $A B C$; so is the Se-
 " mi-diameter of the Earth, $B A$, unto the Moon's Di-
 " stance, $A C$. *Q. E. I.*

Corollary (3.) " From the second Corollary we learn also *Fig. 54.*
 " to measure the Distance of the Sun, for there being given
 " by Astronomical Observations the Angle of the Menstrual
 " Parallax, (namely, that which is made when the Moon
 " appears precisely bisected) or the Angle $Z E O$, and, to-
 " gether with this Angle, the Moon's Distance, $Z O$. We
 " find the Distance of the Sun by this Analogy. As the
 " Sine of the Angle $Z E O$, is to the Sine of the Angle
 " $E O Z$; which Sine is the Radius: So is $Z O$, the Moon's
 " Distance, unto $Z E$, the Distance of the Sun. *Q. E. I.*

P R O P. XXI. Theorem.

THE Angles ($B Q C$, $B F C$) which in a Cir- *Fig. 33.*
 cle stand upon the same Arch ($B O C$), or
 which are in the same Segment ($B Q S C$) are all
 equal among themselves.

Let first the Segment $B Q S C$ be greater than a Semi-
 circle. From the Center A draw $A B$, $A C$. By the fore-
 going, the Angle $B A C$, at the Center, is double of each,
 $B Q C$, $B F C$. Therefore they all, $B Q C$, $B F C$, are
 equal (*a*). *Q. E. D.* (a) *Per*
 Then let the Segment $B Q C$ be equal to, or less than a *Axiom 6.*
 Semi-circle. In the Triangles $B Q I$, $C F I$, because the *Fig. 34.*
 Angles vertically opposite at I are equal (*b*), the sum of (*b*) *Per 15.*
 the rest, Q and R will be equal to the Sum of the (*c*) rest, *l. 1.*
 F and O . Wherefore, if from these equal Sums there be (*c*) *Per*
 taken away the Angles R and O , which by the first Part, *Corol. 10.*
 are equal, as standing upon the same Arch $Q F$, the Angles *p. 32. l. 1.*
 which remain, $Q F$, must be equal. *Q. E. D.*

Corollary. " Hence we gather in Opticks, that any
 Line $B C$ to the Eye placed where you will in the Cir-
 " cumference

" circumference of the Circle, whereof the Lines is a Chord,
 " appears of the same Magnitude; to wit, because it ap-
 " pears every where under an equal Angle B Q C.

Scholium. " If of two equal Angles, standing upon
 " the same Arch, one of them be at the Circumference,
 " the other also will be at the Circumference.

Fig. 33. 34.

" If it be denied, B Q C shall either be equal to the
 " Angle B I C, on this side the Circumference Q F, or to
 " the Angle B E C, which is beyond the said Circumference.

(d) *Per*

" But the Angle B I C, is (d) greater, and the Angle B E C
 " (d) is less than the Angle B Q C. Therefore, &c.

Corol. 1.

p. 32. l. 1.

PROP. XXII. Theorem.

Fig. 35.

IN any Quadrilateral inscribed in a Circle
 (A B C F) the opposite Angles make two right
 ones.

(a) *Per* 32.

l. 1.

(b) *Per* 21.

l. 3.

Let B F, C A be drawn. The Angle A B C, with the
 (a) two, O and X, make two right Angles. But O is
 equal to I (b), because it stands upon the same Arch, B C:
 And again, X is equal to Z, because it stands upon the same
 Arch, A B. Therefore A B C taken together, with the
 two Angles I and Z, that is, with the whole opposite Angle
 A F C, makes two right Angles. Q E. D.

Corollary (1.) " Hence, if one Side of a Quadrilateral,
 " described in a Circle, be protracted, the external Angle
 " will be equal to the opposite Angle of the Quadrilateral;
 " for the internal, added to either of them, makes two
 " right Angles.

(2.) " Likewise a Circle cannot be described about a
 " Rhombus, because its opposite Angles either fall short of,
 " or exceed two right Angles.

(3.) " Likewise, if in any Quadrilateral A B C F, the
 " opposite Angles F and B are equal to two right ones, a

(a) *Per* 5.

l. 4.

* *Per* 22.

l. 3.

+ *Per* Schol.

p. 21. l. 3.

" Circle may be described about it. For (a) a Circle will
 " pass through any three Angles, C, F, A, and this, so that
 " * fourth be equal to B; which cannot be, unless it doth
 " indeed pass through the Point B †. Therefore it doth
 " pass through it.

PROP.

P R O P. XXIII, XXIV. Theorems.

ARE not necessary; and they treat of similar Segments, which cannot rightly be defin'd without Proportions.

P R O P. XXV. Problem.

To perfect a given Arch (A B C.)

Fig. 36.

Let there be subtended at Random the two right Lines A B, C B; which bisect in I and L. From I and L raise Perpendiculars meeting one another in O. This shall be the Center of that Circle whereof A B C is a Portion.

For (a) the Center is both in the Line I X, and in the (a) *Per* Line L Z. Therefore it is in their common Point O. *Corol. p. 1.*

The Practice. From the Center B, taken in the Arch, *l. 3.* describe a Circle; and with the same Interval, from other Centers in the Arch, describe two other Circles, each of which cuts the former twice. Two right Lines drawn through the Intersections, and crossing each other in O, will give the Center.

P R O P. XXVI, XXVII. Theorems.

I N equal Circles, equal right Lines (C E, F I) Fig. 37. subtend equal Arches; and if the Arches are equal, the Subtenses are also equal.

These two Propositions are plainly Axioms, and need no Demonstration.

Corollary (1.) " If in a Circle A B C D the Arch A B be *Fig. 55.* equal to the Arch D C; A D will be parallel to B C.

" For A C being drawn, the Angles A C B, C A D, as standing on equal Arches, will be equal. Wherefore * *Per 27.* A D is parallel to B C. Q. E. D. *l. 1.*

(2.) " The right Line E F, which is drawn from the *Fig. 56.* Point A, the middle Point of some Arch, and toucheth " the

- “ the Circle, is parallel to the right Line BC , which sub-
 “ tends that Arch. For from the Center D , draw unto
 “ the Point of Contact A the right Line DA , and join DB ,
 “ DC . The Side DG is common, and DB is equal to
 “ DC , and the Angle $BD A$ equal to the Angle $CD A$,
 “ the Arches BA , CA being supposed to be equal. There-
 * Per 4. fore the Angles DGB , DGC are equal *, and conse-
 l. 1. quently are right Angles. But the internal Angles GAE ,
 † Per 18. GAF are also right Angles †. Therefore BC , EF are
 l. 2. parallel *. Q. E. D.
 * Per 28.
 l. 1.

P R O P. XXVIII, XXIX. Theorems.

- Fig. 38. **I***F in equal Circles, the Angles, whether at the Centers (BAC , FLI) or at the Circumference (BOC , FSI) be equal; the Arches also (BXC , FZI) on which they stand are equal; and if the Arches be equal, the Angles also are equal.*

These two Propositions also are plainly Axioms, and need no Demonstration.

P R O P. XXX. Problem.

- Fig. 39. **T***O bisect a given Arch (ABC .)*

Draw AC , which bisect in O . From O draw the Perpendicular OB , meeting the Arch in B . I say the thing is done.

For let AB , BC be join'd. The Sides AO , BO are by the Construction equal to CO , OB ; and the Angles at O are equal as being right ones. Therefore the Bases AB , BC are equal (*a*). Therefore the Arches also (*b*) AB , BC are equal.

- (a) Per 4. BC are equal (*a*). Therefore the Arches also (*b*) AB ,
 l. 1. BC are equal.
 (b) Per 26. The Practice. From the Centers A and C describe with
 l. 3. an equal Interval, Arches cutting each other in the Point
 F and I , the right Line drawn through these Points will
 bisect the Arch ABC .

P R O P.

P R O P. XXXI. Theorem.

THE Angle (BCF) in a Semi-circle, is a right one; that in a Segment greater than a Semi-circle, is less than a right one; that in a Segment less than a Semi-circle, is greater than a right one. Fig. 40.

Part I. From the Center A draw AC . Because AB and AC are equal, the Angles O and B are equal (c). (c) *Per 5.*

For the same Cause the Angles I and F are equal. $\therefore$ 1. Therefore the Angle BCF is equal to B and F together. Seeing (a) therefore the three together make two right Angles, BCF , which is half of two right Angles, is one right Angle. (a) *Per 32.*

Part II. Let the Segment $LOBC$ be greater than a Semi circle, and in it let there be the Angle COL , and let LB , the Diameter of the Circle, be drawn. The Angle COL is less than that $BO L$, which by Part I. is a right one. Therefore, &c.

Part III. Let the Segment LOX be less than the Semi-circle LOB , and XOL be the Angle in it. This will be greater than $BO L$, which is a right one. Therefore, &c.

Corollary ($\therefore$) " Hence we may make a Proof of the Instrument, called a Square, whether it be exactly Rectangular or not. For in what Circle soever the top of the Square is laid upon C , or any Point of the Circumference whatsoever, if the Sides of it do pass through the Points of the Diameter B and F , the Angle is a right one; otherwise not. Fig. 40.

(2.) " If the Sides of a Square be held continually upon the Points B and F , in the mean while that the Angle is moved round, first on one Side, then on the other, the top of the Angle C will describe a Circumference of a Circle, whose Diameter is the Line BF .

(3.) " Hence we learn to raise a Perpendicular at the end of a Line. Let BC be the Line, C the Point given, from whence a Perpendicular is to be raised. From any Point whatsoever A as the Center, let a Circle be described passing through the Point C , and cutting BC in any Point,

" as

- “ as B. If the Diameter B F be drawn, it is manifest that
 “ the Line C F is the Perpendicular required. *Q. E. F.*
Fig. 57. (4.) “ Hence it is manifest, that Circles touching one
 “ another inwardly, do cut all Lines, as A D proportion-
 “ ably; or so, that A C, the Subtense of the lesser, is to
 “ A D, the Subtense of the greater Circle; as A E, the
 “ Diameter of the lesser, is to A B, the Diameter of the
 “ greater. For there being drawn the Subtenses E C, B D,
 “ the Triangles E A C, D A B are equi-angled. For the
 “ Angle A is common, and A C E, A D B are right ones,
 “ as being Angles in a Semi-circle; and therefore A E C,
 “ A B D (*b*) are equal. The Triangles therefore are similar,
 “ by the fourth Proposition of the Sixth Book, and A E:
 “ A B :: A C : A D. *Q. E. D.*
 (b) *Per* “
Corol. 9. “ by the fourth Proposition of the Sixth Book, and A E:
Prop. 32. “ A B :: A C : A D. *Q. E. D.*
l. 1. (5.) “ In a right-angled Triangle B C F, if the Hypo-
Fig. 40. “ thenuse B F be bisected in A, the right Line A C cuts
 “ the Triangle into two equicrural ones, A C B, A C F,
 “ and so a Circle described from the Center A. through
 “ B, must pass through C, the Top of the right Angle.

P R O P. XXXII. Theorem.

- Fig. 42, 43.* **I**F a right Line (C F) touch a Circle, and another (A B) which is drawn from the Point of Contact (A) cut it; the Angle made by the Tangent and the cutting Line, is equal to the Angles which is made in the alternate or opposite Segment.

That is, the Angle C A B will be equal to the Angle L, which is made in the Segment A L B; and the Angle F A B will be equal to the Angle O, which is made in the Segment A O B. For,

- Fig. 42.* First, let the Line A B pass through the Center. Here by *Prop. 18* C A B is a right Angle: And by *Prop. 31.* L is also a right one. Therefore C A B and L are equal.
Fig. 43. Then let the Line A B not pass through the Center. Let the Line A Q therefore be drawn through the Center, and B Q be join'd. Because the Angle in the Semi-circle A B Q (*a*) is a right one, B Q A taken together with B A Q will make one right Angle (*b*). But C A Q is also by *Prop. 18.* of this Book a right Angle. Therefore B Q A with B A Q, are equal to C A Q. The common Angle therefore B A Q being
- (a) *Per* 31.
l. 3.
 (b) *Per* 32.
l. 1.

being taken away, there remains BQA , which is equal to L (c) equal to CAB . Therefore L and CAB are equal: (c) *Per 21*. Which is the first Part to be proved. *l. 3.*

Then FAB and CAB make two right Angles (d) and (d) *Per 13*. in the Quadrilateral $BOAL$, the Opposites L and O make *l. 1.* two right Angles (e). Therefore the two FAB , CAB are (e) *Per 13*. equal to the two O and L . Therefore there being taken *l. 3.* away on one Side CAB , on the other L , which have already been shewed to be equal, there remains FAB equal to O . Which was the other Part to be proved.

PROP. XXXIII. Problem.

UPON a given Line (BC) to make a Segment *Fig. 44.*
of a Circle, in which the Angle shall be equal
to an Angle given.

First, let there be an acute Angle given, ABF ; from B draw BL perpendicular to AB ; and at C , the Extremity of the Line BC , make BCI equal to CBL (by 23. *l. 1*) whose Side shall cut BL in I . From the Center I describe a Circle through B : This Circle will also pass through C inasmuch as because of the Equality of the Angles at B and C , the Sides likewise CI , BI are (by 6. *l. 1*.) equal, and the Segment BQC shall contain an Angle equal to the given one ABF .

For because AB is perpendicular to the Diameter BL , (a) *Per 18*. AB will touch the Circle which BC cuts (a). Therefore *l. 3.* the Angle in the Segment BQC is equal (b) to the Angle (b) *By the foregoing.* ABF .

But if the Angle given be obtuse, as RBC , do as before, and COB will be the Segment required.

PROP. XXXIV. Problem.

FROM a given Circle to take away a Segment, *Fig. 45.*
containing an Angle equal to a given one.

Unto the Diameter of the Circle FA , draw the Perpendicular BAL . Then (e) let AC be drawn, which may (e) *Per 23*. make the Angle BAC equal to that which is given. This *l. 1.* line AC shall cut off the Segment AQC , containing an Angle equal to the given one, as is manifest from *Prop. 32*.

PROP.

PROP. XXXV. Theorem.

Fig. 46, 47, 48. **I**n a Circle two right Lines (*CL*, *BF*) cut one another, the Rectangle *COL* under the Segments of one is equal to the Rectangle (*BOF*) under the Segments of the other. For,

If they intersect each other in *A*, the Center of the Circle, the thing is manifest.

Fig. 46. If one of them *CL* passeth through the Center *A*, and bisects the other *BF*, which doth not pass through the Center; it (*f*) cuts it perpendicularly, and so the Square of *FO* is the same with the Rectangle *FOB*. Let *AL* be drawn. Because *CL* is bisected in *A*, and otherwise divided in *O*.

It will be thus,

(a) *Per 5.* Rect. *COL* } will be equal to *AL* q. (a.)
l. 2. + *AO* q. }

that is, *AF* q.

that is, equal to *AO* q. }
 + *FO* q. } (b.)

(b) *Per 47.*
l. 1. Therefore the common Square *AO* being taken away there will remain

Rect. *COL* equal to *FO* q. that is,
 to the Rect. *FOB*.

Fig. 47. Then if one of the right Lines *CL* passes through the Center, and cuts the other *BF* unequally in *O*, let a right Line drawn from the Center *A* cut *BF* into two equal

(c) *Per 3.* Parts in *I*. In this Case, *AIB* will be a right Angle (c.)
l. 3. Now, because *CL* is bisected in *A*, and otherwise in *O*, it will be thus:

(d) *Per 5.* Rect. *COL* } will be equal to *AL* q. (d) that is,
l. 2. + *AO* q. } *AB* q. that is, to

(e) *Per 47.* *AI* q. } (e.)
l. 1. + *BI* q. }

(f) *By the same.* But *AO* q is equal to *OI* q + *AI* q. (f.) Therefore,

Rect. *COL* } equal to *AI* q. }
 + *OI* q. } + *BI* q. }
 + *AI* q. }

Therefore the common Square *AI* being taken away there remains.

$$\begin{array}{l} \text{Rect. COL} \} = \text{BI q.} \\ + \text{OI q.} \} \end{array}$$

But BI Square is equal to the Rectangle FO B, together with OI Square; (g) because FB is bisected in I, and (g) *Per 5.* otherwise cut in O. Therefore, *l. 2.*

$$\begin{array}{l} \text{Rect. COL} \} \text{ are equal to Rect. FO B} \\ + \text{OI q.} \} \qquad \qquad \qquad + \text{OI q.} \} \end{array}$$

Therefore the common OI q being taken away, there remains,

$$\text{Rect. COL} = \text{Rect. FO B.}$$

But lastly, if neither of the Lines CL, FB passes through the Center: Through their common Intersection (O) let there be drawn the right Line XZ, which passes through the Center (A). By what hath been just now demonstrated, both the Rectangle COL, and that FO B, are equal to the Rectangle ZO X. Therefore COL, FO B, are equal betwixt themselves (b).

[“ Or the Proposition may be demonstrated more easily (h) *Per Axiom 1.* and universally thus. Join AC and BD. Here, because of the Equality of the Angles CEA, BED, as (a) *Per 15.* being vertically opposite (a); and of the Angles C and D, as being upon the same Arch AD†: The Triangles CEA, BED are equi-angled (per *Corol. 9. p. 32. l. 1.*)† *Per 21.* Therefore * CE : EA :: EB : ED. Therefore CE × ED is equal to EA × EB (per 16. *l. 6.*) Q. E. D. *l. 6.*

PROP. XXXVI. Theorem.

From (B) a Point given without a Circle, Fig. 49, 50; there be drawn unto the Circle two right Lines, 51, (BF) touching it, the other (BC) cutting it; the Rectangle (CBO,) which is comprehended under the whole cutting Line (CB) and the Part (BO,) which lies betwixt the Point (B) and the Circle, is equal to the Square of the Tangent (BF).

1. If the cutting Line BC passes through the Center A, in AF. This, with the Line FB, will make a right Angle (a). And therefore, because CO is bisected in A, and (a) *Per 18.* it is added OB; it will be thus; *l. 3.*

$$\begin{array}{l} \text{Rect. CBO} \} \text{ will be equal to AB q (b) that is, (b) } \text{Per 6.} \\ + \text{AO q} \} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{Per 2.} \end{array}$$

G

$$\begin{array}{l} \text{to AF q} \} \\ + \text{FB q} \} \text{ (c) } \text{Per 47.} \\ \qquad \qquad \qquad \text{Per 1.} \end{array}$$

Therefore the equal Squares $A O q$, $A F q$ being taken away on both Sides, there remains,

$$\text{Rect. } C B O = B F q.$$

Fig. 50, 51. 2. But then, if $C B$ doth not pass through the Centre let there be drawn $A B$, $A F$, $A O$ and $A L$, and let A bisect $O C$ in L . The Angle $A L O$ is therefore a right

(d) *Per 3.* one (d). Likewise $A F B$ is a right Angle (e). Now, because $C O$ is bisected in L , and to it is added $O B$, it will be thus:

$$\begin{array}{l} l. 3. \quad \text{Rect. } C B O \} = L B q \text{ (f.)} \\ \quad \quad \quad + L O q \} \end{array}$$

(f) *Per 6.*

l. 2. Let there be added on both Sides $A L$ Square, and the

$$\begin{array}{l} \text{Rect. } C B O \} \text{ equal to } L B q. \} \\ \quad \quad \quad + L O q. \} \quad \quad \quad + A L q \} \\ \quad \quad \quad = A L q. \} \end{array}$$

(g) *Per 47.* But the Squares of $L O$, $A L$ are equal (g) to the Square of $A O$, or $A F$; and the Squares of $L B$, $A L$ are equal

(h) *By the* the Square of $A B$ (h.) Therefore,

$$\text{Rect. } C B O \} = A B q. \text{ that is, (i)}$$

(i) *By the same.*

$$+ A F q. \}$$

$$\begin{array}{l} \text{to } B F q. \} \\ + A F q. \} \end{array}$$

Therefore the common Square, that of $A F$ being taken away, there remains

$$\text{Rect. } C B O \text{ equal to the Square of } B F. \quad \text{Q. E. D.}$$

Fig. 59. [" Or more easily and universally thus. Draw $A B$

* *Per 32.* " and $B C$. Now because of the Equality of the Angles

l. 3. " A . and $D B C$, * and for that the Angle D is common

+ *Per Corol.* " both; the Triangles $B D C$, $A D B$ are equi-angled

9.p. 32.*l. 1.* " And therefore (by *Lib. 4. 6.*) $A D : D B :: B D : D C$

* *Per 17.* " Wherefore the Rectangle * $A D \times D C$ is equal to the

l. 6. " Rectangle $B D \times D B$, or $D B q$. Q. E. D.]

Corollaries.

Fig. 52.

1. IF from the same Point B , without the Circle, as many cutting Lines $B C$ as you will be drawn, all the Rectangles $C B O$ are equal amongst themselves. For each of them is equal to the Square of the Tangent $B F$.

2. The right Lines $B F$, $B Q$, which, from the same Point, touch the Circle, are equal. For each of the Squares is equal to the same Rectangle.

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3. " It is also clear, that from the same Point B, taken without the Circle, there can only two Lines B F, B Q be drawn, which shall touch the Circle. For if a third be said to touch it, it must be equal to B F, or B Q, and therefore the same with one of them.

4. " In every Right-angled Triangle B F A, that is not *Fig. 44.* also an Isosceles, the Rectangle arising from the Sum of the Hypotenuse, and one Side drawn into the Difference betwixt them, is equal to the Square of the other Side. For the Sum of the Hypotenuse B A, + A F or A C, is = B C. And their Difference is B A - A F = B A - A O = B O. And the other Side of the Triangle is B F. But the Rectangle C B O is equal to the Square of B F. Therefore, &c.

PROP. XXXVII. Theorem.

If the Rectangle under C B and O B be equal Fig. 52.
to the Square of B F, this must touch the Circle in F.

From B let there be drawn the Tangent B Q, and the Lines E Q, E F being drawn from the Center E, unto the Points Q and F, let B E be joined. Because by the Proposition the Square of B F is equal to the Rectangle C B O, as is also the Square of B Q, by 36th of this Book, Squares of B Q, B F shall be equal betwixt themselves, consequently the right Lines B Q, B F are equal. Therefore the Triangles F E B, B E Q are Equilateral each other. Therefore the Angles Q, F are equal (a.) But (a) *Per 8.* is a right Angle (*per 18. l. 3.*) Therefore F also is a right Angle. Therefore B F toucheth the Circle (b). (b) *Per 16. l. 3.*

Corollaries (1.) " Hence the Angle E B F is equal to the Angle E B Q (*per 8. l. 1.*)

(2.) " If two equal right Lines B F, B Q fall from some Point B upon the convex Circumference, and B F, one of them toucheth the Circle, the other B Q must touch (a) *By the foregoing.* For seeing B F, B Q are equal, their Squares are also equal. But B F q is equal to C B O (a). Therefore B Q q = C B O (b). Therefore B Q also toucheth the Circle (c). *Axiom 1.* (c) *By this Proposition*

3. "

G 2

Scholium

Scholium [1] " Seeing all Planes passing through the
 " Center of the Earth, which all stand perpendicular upon
 " the Horizon, do produce great and equal Circles upon
 " the Earth's Surface, we shall here bring in some elegant
 " Conjectures from thence, out of our Author in his
 " Astronomy; which from the Nature of Circles may very
 " easily be understood.

(1.) " If in any Part the Surface of the Earth were perfectly
 " plane, Men could no more stand upright upon it
 " than upon the side of an Hill, saving in the Point of
 " Contact only.

(2.) " The Head of a Traveller performs a longer Way
 " or Course than his Feet: Likewise he that is on Horse
 " back, and goes the same Way as a Footman, measures
 " a greater or longer Space than he that is on Foot.
 " likewise in a Ship, the uppermost Part of the Mast
 " covers more Way than the lower Parts of it.

(3.) " If any one should travel over the whole Circumference
 " of the Earth, the Way gone over by his Head
 " would exceed that which was gone over by his Feet,
 " the Difference of Circumferences, or by the Circumference
 " of a Circle, whose Semi-diameter is the man's
 " Stature.

(4.) " If a Vessel full of Water be elevated perpendicu-
 " larly, the Water will continually be running over, and
 " it will remain full; namely, because the Surface of the
 " Water is continually compressed into the Surface of a
 " greater Sphere. Yea, if a Vessel be elevated continually
 " higher and higher, the Surface of the Water which
 " is contained in it, will continually descend and come near
 " unto a Plane; unto which yet it will never actually come.

(5.) " If a Vessel full of Water be carried directly down-
 " wards, although nothing run over, yet it will continually
 " be full; namely, because the Surface of the Water falls
 " continually into a Part of a lesser Sphere. From whence
 " it follows.

(6.) " That one and the same Vessel contains more Water
 " at the Foot of a Mountain than at the Top; as it does
 " likewise more in a subterraneous Cellar, than in a Chamber.
 " To which Things add,

(7.) " That two Strings, on which two Iron Balls hang
 " perpendicular, [and consequently the Walls of Buildings
 " erected perpendicularly] are not Parallel one to another.

but Parts of Radius's meeting together in the Center of the Earth.

Scholium [2.] " I think it not amiss to insert in this *Fig. 60.* Place the following Problem also, which was communicated to me by a Friend, as demonstrated by me somewhat more briefly.

" Through the two Points (B) and (C) in a given Circle (F D M) to draw the Circumference of a Circle which shall bisect the Circumference of the other given Circle.

" Through the Center A, and one of the given Points B, let there be drawn the infinite right Line B A M E. Unto which, from the Center, let there be erected the Perpendicular A D, and let B D be drawn. Let the Line D E, be made perpendicular to B D, cutting the infinite Line B A M E in the Point E. Lastly, let a Circle be drawn (a) through the three Points, B, C, E. I(a) *Per 5.* say the Thing is done. For, *l. 4.*

" Let the Chord of the second Circle be drawn through either of the Intersections of the Circles, as G, and through A the Center of the first Circle, to wit, G A f; let also the Diameter of the first Circle G A F be drawn. Then in the first Circle (by *Corol 1. Prop. 8. l. 6.* and by *Prop. 17. l. 6.*) $BA \times AE = AD^2$, that is, (because of the Equality of the Semi-diameters. A D, A G, A F) $= A, G \times A F$. And in the second Circle there will be (b) $AB \times AE = AG \times Af$. Therefore $AF = Af$, and (b) *Per 35.* the Points F and f will coincide, and the Arch F D G is *l. 3.* equal to the Arch F M G. Q. E. F.





T H E
Elements of EUCLID.
BOOK IV.

THIS Book, which is wholly Problematical, teacheth by what Artifice, Figures, those which are ordinate or regular especially, may be inscribed in, and circumscribed about Circles. There is very great Use of it in building Fortifications; and from it, as a Fountain, have been derived those most excellent Tables of Sines, Tangents and Secants, to the very great Benefit of the Mathematicks,

[“ This Book is most useful for *Trigonometry*; for by inscribing *Polygons* in a Circle, we learn to frame Tables of *Chords*, *Tangents* and *Secants*; by the help of which we learn to measure the *Magnitudes* of *Figures* and *Bodies*. Neither without it can we duly *distinguish* the *Aspects*, as they call them, of the *Stars*, as the *Quartile*, *Sextile*, &c. they wholly depending upon the *Inscriptions* of *Polygons* in *Circles*. Neither can we otherwise collect the *Area* (which is a certain *Quadrature* of a *Circle*) than from the *Area*’s or *Squares* of innumerable *Polygons* inscrib’d in, and circumscrib’d about a *Circle*. And in like manner we know the duplicate *Proportion* of *Circles* amongst themselves, from the duplicate *Proportion* of *Polygons* inscribed in, or circumscribed about *Circles*. And as for *Military Architecture*, it makes so much use of *Polygons* inscrib’d in *Circles*, that more than all other Sciences it may seem to be wholly owing to this Book.]

De-

DEFINITIONS.

A Rectilinear Figure is said to be inscrib'd in a Circle, or to have a Circle circumscrib'd about it, when the Tops of all the Angles thereof are in the Circumference of the Circle.

2. A Rectilinear Figure is said to be circumscrib'd about a Circle, or to have a Circle inscrib'd in it, when each one of its Sides toucheth the Circle.

3. An ordinate or regular Figure is that which is Equilateral and Equiangular.

PROPOSITION I. Problem.

To inscribe a right Line (A,) which is not greater than the Diameter, into a Circle (BD.) Fig. 1. l. 4.

Take in the Circumference any Point B. From the Center B, with the Interval of the given Line A, describe an Arch, cutting the Circle in C. Draw the right Line BC. I say the Thing is done.

PROP. II. Problem.

To inscribe in a Circle a Triangle having equal Angles with a given one (X). Fig. 2.

Let the Line EF touch the Circle in D. Let EDG be made (a) equal to the Angle C, and FDH equal to B; and join GH. I say the Thing is done. For (b) EDG is equal to H. H consequently is equal to the Angle C. And FDH is equal (d) to G; and consequently G is equal to B. Therefore GDH (e) is equal to the Angle A. therefore what was required is done.

Construc-
tion.

(d) Per 23.
l. 3.

(e) Per

Corol. 9.
p. 32. l. 1.

Fig. 3.

PROP. III. Problem.

To circumscribe about a Circle a Triangle, having equal Angles with a given one (IKL).

Let the Line IK be drawn forth on both Sides, so as to make the external Angles O and N. Make at the Center the Angles GAB, BAF equal to O, N respectively, which

which is done by 23. *l. 1.* Then in the Points G, F, B, let three right Lines touch the Circle, meeting together in C, E, D. The Triangle C E D is circumscribed about the Circle, and is equi-angled to the given one I L K. For,

- In the Quadrilateral C G A B, the Angles G and B are
 (f) *Per 18.* (f) both of them right ones. Therefore the remaining
 l. 3. ones G A B, and C taken together, do (g) make two right
 (g) *Per* ones, and consequently are equal to the two together, O, L.
Theorem 1. Therefore the two, G A B and O, which are equal by the
Schol. p. 32. Construction, being taken away, there remains C equal to
 l. 1. I. In the same manner E will be proved equal to the Angle
 (h) *Per* gle K. Therefore D and L are (b) likewise equal. That
Corol. 9. therefore is done which was demanded.
Prop. 32. For that the Tangents do concur is thus shewed. The
 l. 1. Angles O, I, and K, N are (a) equal to four right ones
 (a) *Per 13.* and I K are less than two right ones. Therefore O, N
 l. 1. (that is, by the Construction G A B and B A F) are greater
 (b) *Per 32.* than two right ones. Therefore G A F (c) is less than two
 l. 1. right ones. Therefore G F falls between A and D. There-
 (c) *Per* fore seeing A G D, and A F D are right Angles, D G F
Corol. 3. and D F G are less than two right ones. Therefore C G F
Prop. 13. and E F D † meet together towards D. In the same man-
 l. 1. ner it may be demonstrated that the rest concur.

† *Per Schol.*

p. 3. l. 1.

Fig. 3.

PROP. IV. Problem.

TO inscribe a Circle in a Triangle.

Fig. 3.

Bisect the two Angles C and E with the Lines C A, E A meeting together in A. From A draw the Perpendiculars A B, A G, A F. A Circle described from the Center A through B, will pass also through G and F, and touch the three Sides of the Triangle.

For in the Triangles C A G, C A B, because the Angles A G C, A B C, and likewise those G C A, and B C A are equal by the Construction, and the Side A C is common to the Sides A G, A B * must be likewise equal. In like manner A B, A F may be shewn to be equal. Therefore the Circle describ'd from the Center A, passeth through B, G, F. And because the Angles at those three Points

- (f) *Per 16.* are equal, it toucheth (f) all the Sides of the Triangle.
 l. 3. That therefore is done which was required.

[" Hence

[“ Hence the Sides of a Triangle being known, the Segments of them, which are made from the Contacts of an inscribed Circle, will be known. Let DC be 12, DE 18, CE 16, DC and CE will be 28, from which subtract $18 = DE = DG + BE$, there remains $10 = CG + CB$. Therefore CG or $CB = 5$. Consequently EB or $EF = 11$. Wherefore FD or $DG = 7$.]

PROP. V. Problem.

TO describe a Circle about a Triangle, or thro' three given Points, B, C, D , not lying in a right Line, to describe a Circle. *Fig. 4.*

Connect the given Points with two right Lines BC, CD , which bisect with the Perpendiculars EA, OA , meeting together in A . This will be the Center of a Circle which passes through B, C, D .

Let the right Lines AC, AD, AB be drawn. By the Construction, the Sides DO, OA are equal to these, CO, AO ; and the Angles at O are right ones. Therefore AD is equal to AC (*a.*) In the same manner AB may be prov'd equal to AC . Therefore AD, AB are equal. Therefore a Circle described from the Center A through B , will pass also through C and D . Which was the Thing required. *(b) Per 4. l. 1. Axiom 1.*

As for the Practice, it is sufficient to describe from B, C, D , three equal Circles, intersecting each other; and through the Intersections to draw right Lines; these meeting one another will give the Center sought.

PROP. VI, VII. Problems.

TO inscribe a Square in, and circumscribe one about a Circle. *Fig. 5.*

Let the Diameters BD, CE be drawn, cutting each other perpendicularly. The right Lines which join the Terms of these, inscribe a Square in a Circle.

The

The Demonstration is manifest from 4. l. 1. and 31. l. 3. Then let four Tangents be drawn touching the Circle in B, C, D, E, meeting together in I, F, G, H. The Figure I F G H is a Square, circumscrib'd about a Circle.

The Demonstration is manifest from 18. l. 3. with *Corollary 2. Proposition 36. l. 3. and 28, and 34. l. 1.*

Scholium.

Fig. 5. A Square describ'd about a Circle is double to that inscrib'd. For because the Angle B C D in the Semi-circle (c) is a right one, the Square of B D (that is, F I l. 3. Square shall be (d) equal to B C q. + C D q, and therefore (d) Per 47. fore double to the Square of C D, i. e. to C D E B. l. 1.

P R O P. VIII, IX. Problems.

Fig. 6. TO inscribe a Circle in, and circumscribe one about a Square, as (B C F E.)

Let there be drawn the Diameters of the Square, cutting each other in O. From the Center O describe a Circle through B; this will also pass through E, F, C.

Then from the Center O draw O D perpendicular to B C; a Circle describ'd from the Center O through D, will touch all the Sides of the Square.

Part I. Because, by the Hypothesis, the Lines C B, E B (c) Per 5. are equal; the Angles B C E, B E C will be equal (c). But l. 1. C B E is a right Angle by the Hypothesis. B C E therefore (d) Per and B E C are half right ones (d). In the same manner Corol. 11. C B F will be shew'd to be an half right Angle, as likewise pro. 32. l. 1. the rest of the Angles; and so they are equal amongst themselves. Therefore in the Triangle B A C, seeing there are two equal Angles C B O, B C O, the right (e) Per 6. Lines O B and O C (e) are equal. In the like manner the l. 1. right Lines O B, O E, O F may be shew'd to be equal. Therefore a Circle describ'd from the Center O through B, will pass through E, F, C.

Part II. From O let there be also drawn the Perpendiculars O G, O H, O I. Because in the Triangles G B O, D B O, the Angles at D and G, as likewise those at B, are equal, and the

the Side OB is common, the Sides OD , OG must be equal (*a*.) In the same manner OG , OH , OI may be shew'd to be equal. Therefore a Circle describ'd from the Center O , which passeth through D , will also pass through G , H , I , and touch all the Sides of the Square (*b*.) Because the Angles at D , G , H , I , are right ones. Therefore we have done what was required.

PROP. X. Problem.

TO make an Isosceles Triangle BAC , in which the Angle at the Base (ABC , or ACB) shall be double to that which is at the Top (A).

Let any right Line, what you will, as AB , be taken, which so cut in D that the Rectangle ABD shall be equal to AD Square. Then from the Center A , thro' B , describe a Circle; in which inscribe BC equal to AD , and join AC . BAC shall be the Triangle sought.

For let the right Line DC be drawn, and through A , D , C describe a Circle. Because the Rectangle ABD is equal to the Square AD , (that is, BC) it is manifest that BC toucheth the Circle DO , which CD cuts. Therefore the Angle BCD is equal to the Angle A in the opposite Segment; and so the common Angle DCA being added, BCA must be equal to $A + DCA$. But because the Sides AB , AC are equal, ABC is equal to the Angle ACB . Therefore the Angle ABC is also equal to $A + DCA$. But the external Angle also BDC is equal to the two internal ones $A + DCA$. Therefore ABC , and BDC are equal. Therefore the Line DC is equal to BC , that is, by the Construction to DA . Therefore the Angles A and DCA are equal. Wherefore the Angle ABC , which hath been shewed equal to those two, shall be double to one A . That is done therefore which was required.

Corollary.

Corollary.

EACH of the Angles at the Base B and C in the *Isoſceles* now framed, is two Fifths of two right ones, or four Fifths of one right one, and the remaining one A is one Fifth of two right ones, or two Fifths of one right one, And is manifeſt out of this Propoſition taken together with that 32. l. 1.

PROP. XI. Problem.

Fig. 7, 8. **T**O inſcribe a regular Pentagon in a Circle.

(a) *By the foregoing.* Let there be deſcribed (a) the Triangle BAC, having the Angles at the Baſe double to that at the Top. Inſcribe a Triangle CAD equi-angled to this in a Circle (b). Biſect the Angles at the Baſe ACD, ADC, with the right Lines CE, DB, cutting the Circle in E and B. The Points A, B, C, D, E, join'd by right Lines, will give an ordinate Pentagon inſcribed in a Circle.

For from the Conſtruction it appears that the Angles I, N, Q, S, O are equal. Wherefore the Arches ſubtended to them, AE, ED, CD, CB, BA, are alſo equal. Therefore the right Lines ſubtended to thoſe Arches ſhall alſo (d) be equal. The Pentagon therefore is Equilateral. But it is alſo (e) Equiangular, becauſe its Angles BAE, AED, &c. ſtand on equal Arches BCDE, ABCD, &c. that therefore is done which was required.

Corollary.

Fig. 8. **T**HE Angle of a regular Pentagon makes ſix Fifths of one right Angle, or three Fifths of two. For the three Angles at A, ſeeing they are equal, as ſtanding upon equal Arches, BC, CD, DE and the middlemoſt of them by the Corollary foregoing is two Fifths of one right Angle the three together, that is, the Angle of the Pentagon it ſelf muſt make ſix Fifths of one right one.

[Scholium]

[Scholium. " This holds universally, that Figures of an *Fig. 8.*
 " odd Number of Sides are inscribed in a Circle, by means
 " of an *Isoceles* Triangle, whose equal Angles at the Base
 " are multiple of those at the Top. But Figures of an
 " even Number of Sides are inscribed by the means of
 " *Isoceles* Triangles, whose Angles at the Base are each of
 " them multiple sesquialteral of that which is at the Top.
 " As in the *Isoceles* ACD , if the Angle C or D be
 " threefold of A , the Side CD will be the Side of an
 " Heptagon; if fourfold, it will be the Side of an Enne-
 " gon, &c. But if C or D shall be one and a half of A , CD
 " will be the Side of a Square; and if C shall be two and
 " a half of the Angle A , CD will subtend a sixth Part of
 " the Circumference: In like manner, if C or D shall be
 " three and a half of the Angle A , CD shall be the Side
 " of an Octagon, &c.

Scholium.

EUCLID's Inscription of a Pentagon is ingenious, but
 that of *Ptolomy*, which he delivers in the first Book of
 his *Almagest*, is much more expeditious: and it is this:

Let the Diameters ED , BF be drawn, cutting one ano- *Fig. 12.*
 ther perpendicularly in A . Bisection the Radius AD in C .
 From the Center C , through B , describe an Arch, meeting
 the Diameter ED in G . The right Line GB is the Side
 of a Pentagon, and AG of a Decagon.

The Demonstration cannot be given here, for it depends
 upon the 13th Book of *Euclid*. See it in *Clavius*, in his
Scholium, after *Prop. 10. l. 13.*

Problem.

UPON a given right Line (AB) to describe a regular *Fig. 9.*
 Pentagon.

Cut AB so in C (*a*) that the Rectangle ABC may be (*a*) *Per 11.*
 equal to the Square of AD . From AB protracted on both *2.*
 Sides take away AD , BE equal to the greater Segment
 AC . From the Center A and D , with the Interval AB ,
 describe two Arches, cutting each other in F . Likewise,
 from the Centers B and E , describe, with the same Inter-
 val,

val, two Arches cutting each other in G. And again, from the Centers G and F, with the same Interval, describe two others, cutting each other in I. The Points A, F, I, G, B, being joined, will give a regular Polygon upon the right Line A B.

That it is Equilateral, is manifest from the Construction; that it is equi-angled, will be thus demonstrated. Let D F be drawn. It is manifest by the Construction, that A D F is an *Isoceles*. And the Base A D is the greater Segment of the Side D F, so divided, that the Rectangle of the whole and the lesser Side, is equal to the Square of the greater. (For D F is equal to A B, and A D equal to A C.) Therefore the Angle D A F is two Fifths of two right ones; by *Corol. Prop. 10. l. 4.* Therefore the remaining Angle F A B is three Fifths of two right ones, or six Fifths of one right one (b); and therefore is an Angle of a regular Pentagon (c). In the same manner may it be shewn, that the Angle G B A is three Fifths of two right ones, and so equal to F A B. From whence it is necessary, that the rest F, G, I, should be equal to these, as appears from their being Equilateral to these, if the right Line F G be conceived to be subtended, as appears by *Prop. 8. l. 1.*

(b) *Per* 13.
l. 1.
(c) *Per* *Co-*
rol. 5. p. 11.
l. 4.

P R O P. XII. Problem.

Fig. 10. **T**O circumscribe an ordinate Pentagon about a Circle.

Let there, by the foregoing, be inscribed the regular Pentagon G H I K M, and let there be drawn Tangents in the Points G, H, I, K, M, which may concur in B, C, D, E, F. I say the Thing is done.

For from the Center draw the right Lines, A G, A B, A H, A C, A I. Here, because from the same Point B, B G, and B H touch the Circle, they (a) are equal. Therefore the Triangles G A B, B A H are Equilateral to each other. Therefore (b) the Angles O, P, as likewise those Q, S, are equal. And therefore the whole Angle B is double to P, and the whole G A H double to S. For the same reason the Angles C and H A I are double to T and N respectively. But G A H and H A I are equal (c), because they stand upon equal Arches, by Construction G H, H I. Therefore their halves, S and N are also equal.

(a) *Per*
Corol. 2. p.
36. l. 3.
(b) *Per* 8.
l. 1.
(c) *Per* 29.
l. 3.

equal. Because therefore in the Triangles $B A H$, $H A C$, the two Angles S and N are equal, and those at H are both right Angles (d), and likewise the Side $A H$ is common; (d) *Per* 18. therefore the Sides (e) $B H$, $C H$, as likewise the Angles P , T , are equal. In the same manner I might shew $B G$, (e) *Per* 26. $F G$ to be equal. Therefore $B F$, $C B$, which are double $F G$, to the Equals $B G$, $B H$, are also equal. In the same manner it may be shewed that the rest of the Sides of the circumscribed Pentagon are equal. It is therefore Equilateral; but it is also equi-angled; for seeing it hath been shewed that the Angles B and C are each of them double to the Equals P and T , they must also be equal betwixt themselves. And in the same manner of the rest. We have therefore described a regular Pentagon about a Circle. Which was the Thing to be done.

In the same way any ordinate Figure whatsoever is described about a Circle, that is, if a like Figure be first inscribed in the Circle.

PRO P. XIII, XIV. Problems.

To inscribe a Circle in a regular Pentagon, and Fig. II. circumscribe one about it.

Bisect the two Angles of the Pentagon B , C , with the right Lines $B N$, $C S$, cutting each other in A . From A draw the Perpendicular $A L$.

A Circle described from the Point A , with the Interval $A L$, touches all the Sides of the Pentagon; and a Circle described from the same Point A , with the Interval $A B$, passes also through the Points E , F , D , C .

Part I. In the Triangles $D C A$, $B C A$, because the Sides $D C$, $C A$, are equal to $B C$, $C A$, by the Hypothesis, and the Angles P and O are equal by the Construction, those also G and I will be equal by 4. $\therefore$ 1. Now, the whole also B and D are equal by the Hypothesis. Wherefore being the Angle G is half of B by the Construction, I also will be half of D . Therefore D is bisected by the right Line $D M$. For the same Cause the rest of the Angles of the Pentagon $E F$, are bisected, and consequently the half Angles are equal among themselves. Now, let the Perpendiculars be drawn, $A M$, $A S$, $A N$, $A R$. For because

because in the Triangles LBA , MBA , the Angles G and BLA are equal to the Angles Q and BMA , by the Construction and the Side BA is common, AL and AM must be also equal (*a*). In like manner I might shew that the rest of the Perpendiculars, AM , AN , AS , AR , are equal. A Circle therefore from the Center A , passing through L , will likewise pass through M , S , N , R , and because the Angles at L , M , S , N , R , are right ones by the Construction, * it will touch the five Sides of the Pentagon. Which was the first Part.

Part II. In the Triangle CAB , because the Angles C and G have already been shewn to be equal, the Sides also AC , AB must be equal (*b*), and in the same manner, AB , AF , AE , AD , may be proved equal, and therefore a Circle from the Center A passing through B , must pass also through C , D , E , F . Therefore we have both inscribed a Circle in a Pentagon, and circumscribed one about a Pentagon. *Q. E. D.*

[“ In the same way, in any regular Figure whatsoever a Circle may be inscribed, and circumscribed about it.

PROP. XV. Problem.

Fig. 13. IN a given Circle to describe a regular Hexagon.

Let the Diameter FAB be drawn. From the Center A through A , describe a Circle, cutting the given one in C and D . Likewise from the Center F , through A , a Circle cutting the given one in E and G . The six Points, B , C , E , F , G , D , connected by right Lines, will give the Hexagon required.

From the Center A , let fall the right Lines AE , AC , AG , AD . It is manifest that the Triangles H , I , M , N are Equilateral, both in themselves, and with one another. (*c*). Then because the Angles CAB , EAF , each of them make one Third of two right Angles (*per Corollary* *p. 32. l. 1.*) and therefore do make both together two Thirds of two right Angles; it remains (*d*) that EAC be one Third of two right Angles; therefore the Angles EAC , CAB are equal. But the Sides also EA , AC are equal to the Sides BA , AC . Therefore the Base EC

er 4. *l. 1.*) is equal to the Base B C, that is to the Radius A C by the Construction. Wherefore the Triangle N also Equilateral. And in the same manner the Triangle may be shewn to be so. Because therefore all the six Triangles H, I, K, L, M, N, are Equilateral; it is manifest that all the Sides, C B, B D, D G, G F, F E, E C, are equal one to another, and to the Radius A C. The Hexagon is therefore Equilateral. But it is also Equiangular, being each one of its Angles E C, B D, G F, consists of two Angles of an Equilateral Triangle. Therefore we have described a regular Hexagon in the Circle.

Corollaries.

THE Side of an Hexagon inscribed in a Circle, [or a Chord of 60 Degrees] is equal to the Radius, [and consequently the Sine of 30 Degrees is equal to half the Radius, *per Corollary 2. p. 3. l. 3.*]

2. An Angle of a regular Hexagon is four Thirds of one right Angle; as consisting of two Angles of an Equilateral Triangle, each of which makes two Thirds of a right Angle.

3. If there be drawn the Diameter P S, perpendicular to the other F B; and with the Interval of the Radius P A, from the Center P and S, there may be made Sections in Q, in R and T, and in like manner from the Centers F and B, make the Sections in G and E, in D and C, the Points P, E, O, F, R, G, S, D, T, B, Q, C, connected with right Lines, will give a Figure of twelve Sides, inscribed in a Circle with one Aperture of the Compasses, which Thing is of great Service in Dialling.

4. From what has been demonstrated, we may easily describe an Equilateral Triangle in a Circle. The Diameter B being drawn, from the Center B, through A, describe the Arch C A D. The Points C, F, D, connected with right Lines, will give the Triangle sought.

5. The Side C X D of the Equilateral Triangle, cuts off from the Diameter B F perpendicular to it, a fourth Part thereof B X. For the Angles A C X, B C X, standing upon equal Arches G D, D B are equal (*per 29. l. 3.*) and the Sides A C, C X are equal to the Sides B C, C X. Therefore A X, B X are equal (*o*). Therefore B X is the fourth (*a*) *Per 4. l. 1.*

H

Scolium.

Scholium 1. Problem.

Fig. 13. **Y**OU may raise a regular Hexagon upon a right Line BC thus. Make an * Equilateral Triangle CA upon the given Line CB. From the Center A through and C describe a Circle. This will contain an Hexagon upon the given right Line CB. The Thing is manifest from the Proposition, and *Corollary 1.*

Theorem.

THE Square of a Side of an Equilateral Triangle is treble to the Square of the Semi-diameter of a Circle in which it is inscribed, and is to the Square of the whole Diameter, as 3 to 4.

Fig. 14. Let there be drawn the Semi-diameter AD. The Square of FD is equal to FA q + DA q + the Rectangle FAX twice taken (*per 12. l. 2.*) But the Rectangle FAX twice taken, is equal to the Square of the Semi-diameter FA q DA: (For because AX, XB (*b*) are equal, the Rectangle FAX twice taken, is equal to the two Rectangles which are under FA, AX, and under FA and XB, that is, equal to the Rectangle FAB (*c*); that is, equal to FA q. Therefore FD q is treble to FA q or DA q, the Square of the Semi-diameter.

Now, because the Square of the whole Diameter is Quadruple of the Square of FA, the Semi-diameter (*d*), it is manifest that the Square of FD is to the Square of the Diameter, as 3 to 4.

Prop. 4. l. 2. Hence it follows, that a Side of an Equilateral Triangle is to the Diameter, as the Square Root of 3 is to 2, the Square Root of 4; and therefore that those Lines are incommensurable.

PROP. XVI. Problem.

To inscribe a regular Quindecagon in a Circle. *Fig. 15.*

Inscribe the Circle A C, the Side of a Pentagon (*a*), and (*a*) *Per* 11; D, the Side of an Equilateral Triangle (*per* Corol. 4. *l.* 4. 15. *l.* 4.) bisect the Arch C D in E. C E is the Side of the Quindecagon, or fifteen-angled Figure sought. For if the whole Circumference be supposed to be 15, the Arch A C will be 3, and the Arch A D 5, and therefore the Arch C D 2, and consequently C E 1.

Corollary.

BY this Method innumerable regular Figures may be inscribed in a Circle. For if A C, A D, the Sides of two regular Figures be inscribed in a Circle, the Difference of the Arches C D will contain so many Sides of a new regular Figure, as are the Units whereby the Denominators of the former differ one from another. But the Denominator of the new Figure is had, if the Denominators of the former multiplied one by the other. As if A D be the Side of a Square, and A C of a Decagon, the Difference of the Denominators is 6. Therefore the Arch C D contains 6 Sides of a new Figure. But the new Figure is of 40 Sides. For the Denominators 4 and 10 multiplied one by the other, make 40.

Scholium.

THERE hath not yet been found out the Art by which regular Figures of 7, 9, 11, 13, 17, &c. Sides may be inscribed in a Circle, by a Pair of Compasses and a Rule only; forasmuch as that Inscription of Figures depends on the Division of the Circumference into any given Parts, which thing is lacking: But if the Circumference of a Circle be divided into 360 Parts, you may, in a mechanical Way, inscribe any regular Figures whatsoever, in it, in this manner.

Problem 1.

Fig. 15. **D**IVIDE 360 Degrees (that is, the whole Circumference) by the Denominator of the Polygon to be inscribed (*e. g.* a Nonangle.) Make at the Center the Angle A G of so many Degrees as are the Units of the Quotient in said Division. A K shall be the Side of the nine angled Figure, which is required to be inscribed in the Circle.

Problem 2.

Fig. 15. **B**UT upon a given right Line you may describe a regular Figure whatsoever by the help of the following Table:

A right Angle is to the Angle of the Figure.

Difference.

In a Pentagon, as	-	-	-	5	to	6—1
In an Hexagon, as	-	-	-	3	to	4—1
In an Heptagon, as	-	-	-	7	to	10—3
In an Octagon, as	-	-	-	2	to	3—1
In a Nonagon, as	-	-	-	9	to	14—5
In a Decagon, as	-	-	-	5	to	8—3
In an Undecagon, as	-	-	-	11	to	18—7
In a Duodecagon, as	-	-	-	3	to	5—2

Fig. 15. Let a regular Heptagon be to be inscribed upon a given right Line X B. From the Center X, with the interval X B, describe a Circle, from which cut off the Quadrant B O. See in the Table what is the Proportion of a right Angle, to the Angle of an Heptagon: You will find it to be as 7 to 10, and the Difference is 3. Divide the Quadrant therefore into seven equal Arches, so many which add to it from O to N, as the Difference hath Units. Through three Points, B, X, N, describe (*per* 5. *l.* 4.) a Circle. This contains an Heptagon of the given right Line X B.

The Table was made by means of *Theorem II.* in the *Schol.* upon *p. 32. l. 1.* by which is found the Number of right Angles, which the Angles of any right-lin'd Figure make; which Number being divided by the Denominator of the Figure, gives the Denominator of the Proportion of the Angle of the Figure to a right one.

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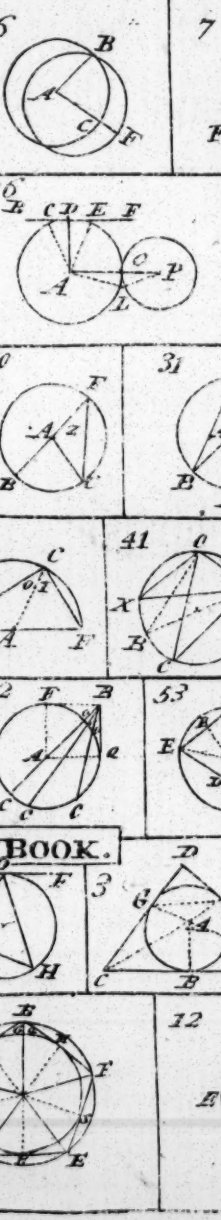
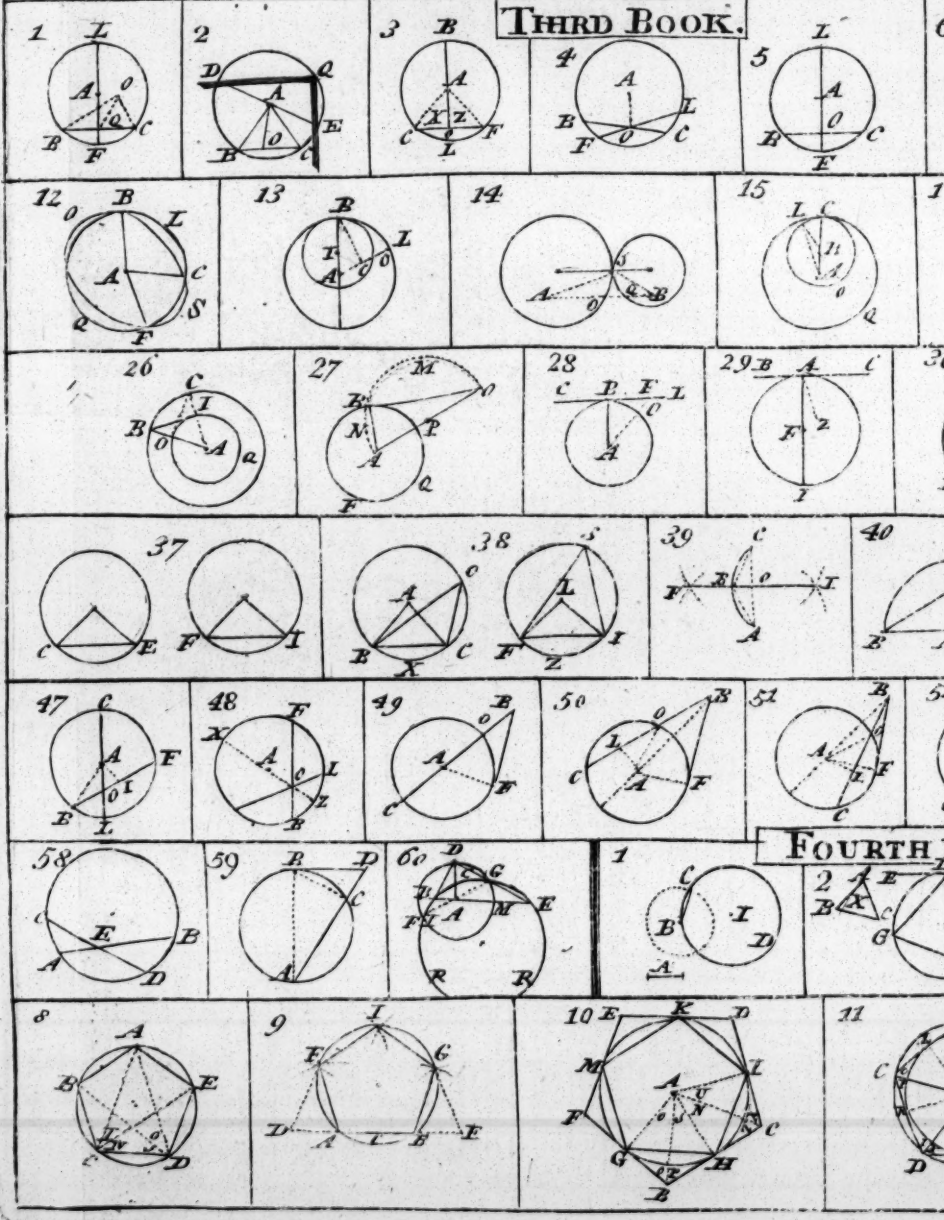
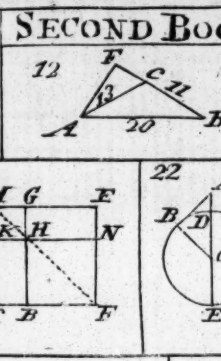
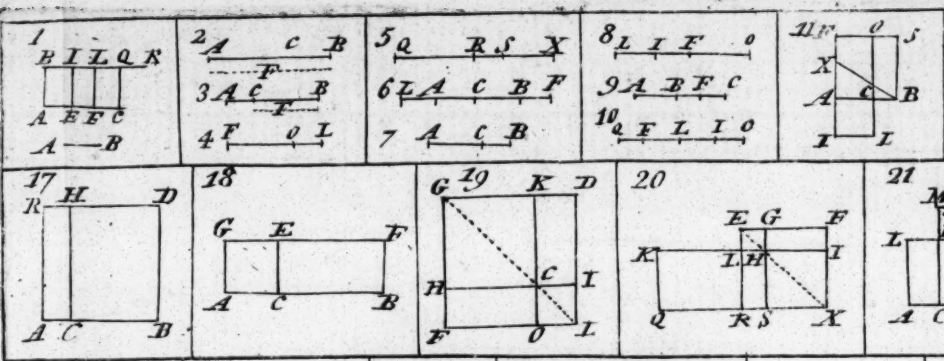
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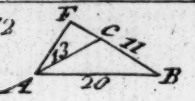
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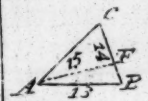
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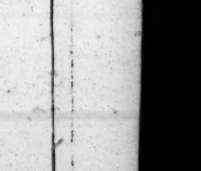
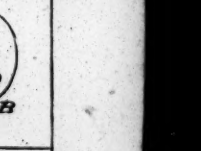
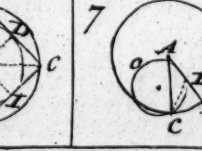
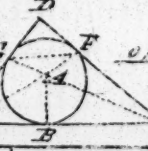
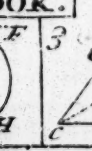
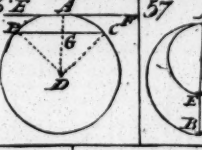
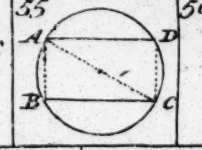
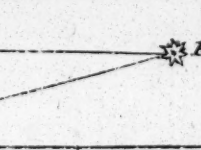
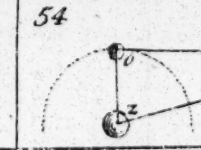
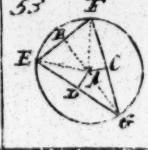
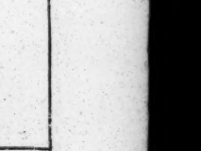
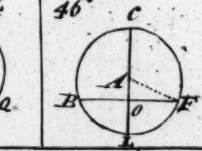
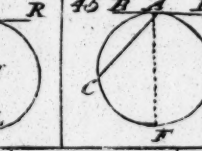
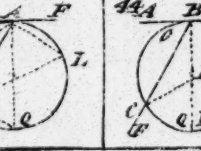
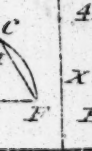
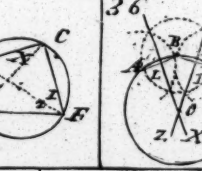
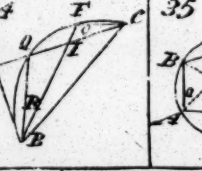
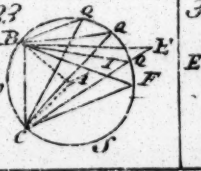
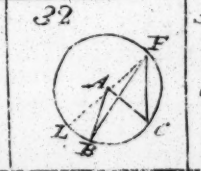
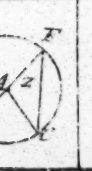
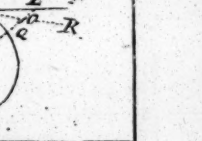
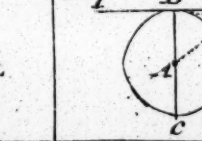
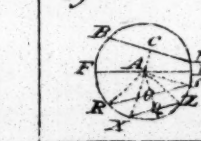
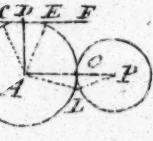
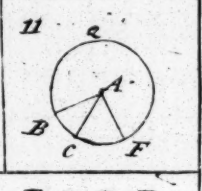
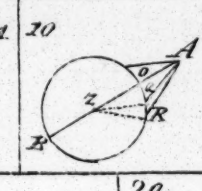
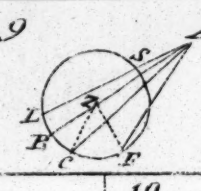
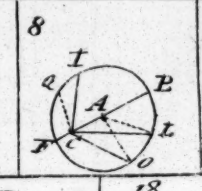
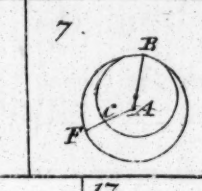
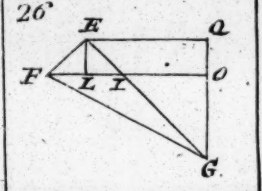
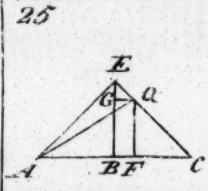
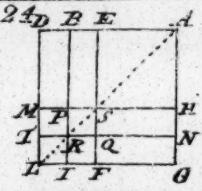
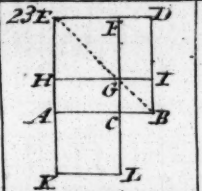
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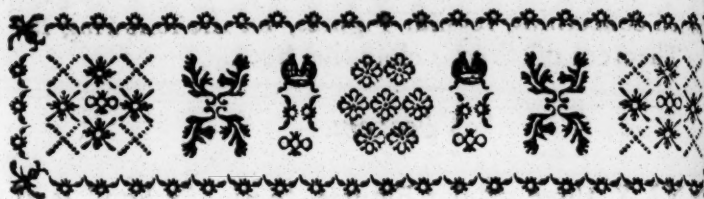
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Now, because hitherto many Things have been propounded concerning regular Figures, let the following famous Theorem of *Proclus* close this Book.

Theorem.

ONLY three regular Figures, to wit, six Equilateral Triangles, four Squares, and three Hexagons, can fill Space; that is, constitute one continued Superficies. which is thus demonstrated. That some regular Figure, *Fig. 13.* when repeated, should be able to fill a Space; it is required that the Angles of many Figures of that kind being disposed about one Point, should make just four right ones; for just many right Angles may be placed about one Point, as appears from *Corollary 3. Prop. 13. l. 1.* As for Example, that Equilateral Triangles should fill a Space, it is required that so many Angles of such Triangles N, M, L, K, I, H, being disposed about the Point A, should make just four right ones. But six Angles of an Equilateral Triangle do make four right ones; (for one makes two Thirds of one right one *, and therefore six of them make twelve Thirds * *Corol. 12. p. 32. l. 1.* one right one, that is, four right ones :) Likewise the Angles of a Square make four right ones, as is manifest; likewise three Angles of an Hexagon; for one makes four Thirds of one right Angle, (*per Corollary 2. p. 15.*); and therefore three of them do make twelve Thirds one right Angle, that is, four right ones. Therefore, But that no other Figure beside these can do this, will manifestly appear, if its Angle being found, as above, you multiply the same by any Number whatsoever; for the Angles will always either fall short of, or exceed four right ones.



T H E

Elements of EUCLID

B O O K V.

THIS fifth Book of Elements is altogether necessary for demonstrating the Propositions of the first Book. The Doctrine which it containeth is almost continual Use. The Way of reasoning from Geometrical Proportion is most subtil, solid and brief. This Method of Reasoning, as a kind of Mathematical Logic, Geometry, Arithmetick, Musick, Astronomy, Statics, and all the other Parts of the Mathematicks, make especial Use of: Forasmuch as they almost wholly depend upon Proportions connected together one with another; and will want to borrow their Ways of reasoning concerning Proportionals from this fifth Book. Practical Geometry, which consists in the measuring of Lines, Figures and Solids, is for the most part derived from the Doctrine of Proportions. There is not a Rule in Arithmetick, but what may be demonstrated from the Propositions of this fifth Book, without the help of the 7th, 8th, and 9th Books, which treat professedly of Numbers. We may fitly call the Musick the Antients, Geometrical Proportions applied to tune Sounds; which same Thing you may well nigh say concerning Statics, which are conversant about the Weights of Bodies. To comprehend the whole Matter in few Words: If you take away the Doctrine of Proportion from the Mathematicks, you will leave almost nothing which is excellent, or greatly to be accounted of.

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Scholium.

" There is no Mathematician who is ignorant of how great Importance in Geometry the Knowledge of Proportions is ; for it is the very Marrow, as it were, of the Mathematical Sciences ; And the various Ways of Reasoning concerning Proportionals, are both most useful, and most certain ; neither can we without them move one Step.

" But then I reckon that this Doctrine is congenite in Men's Minds with common Reason itself ; and that the various Ways of Reasoning concerning Proportionals, which *Euclid*, by much winding and going about, delivers in this whole Book, do not so much need Demonstration, properly so called, as Illustration and Examples. And I am altogether of Opinion, that those who take in hand to deliver this most easy Doctrine by a long Circuit of Propositions, do involve a Thing in itself most clear, in a certain Cloud, and render it far more difficult. The Sum of the Matter I will open in a few Words. It is a thing easily known, that four Quantities are then proportional, or that the Analogies are then alike, when the first Quantity contains the second, as often as the third contains the fourth ; or when the first is as often contained by the second, as the third is by the fourth. So $16 : 8 :: 4 : 2$. And $3 : 9 :: 4 : 12$. Here are like, or the same Proportions ; because in the former Example, the Consequents 8 and 2 are contained twice in their respective Antecedents ; and so the Proportion of the Antecedents to the Consequents, is double. And in the other Example, the Proportions are also alike, because the Consequents 9 and 12 do contain their respective Antecedents three times ; and so the Proportion of the Antecedents, to the Consequents, is sub-triple. (Nor is there any Proportion of commensurable Quantities which may not be expressed by certain Numbers ; nor indeed of incommensurables, which may not be expressed by Numbers infinitely approaching nearer and nearer unto the true one.) Furthermore, from what hath been said it appears, that like Proportions, whatsoever they are, may be expressed not only by divers Numbers, but also by the same. Thus 2 to 1 designs as well the Propor-

" tion

" tion of 16 to 8, as of 4 to 2; 1 to 3 no less expressed
 " that of 4 to 12, than that of 3 to 9, as is most manifest
 " Supposing therefore four Quantities to be proportional
 " $A : B :: a : b$; it is enquired in this Book after how
 " many like Manners the Terms of these like Proportions
 " may be changed, and ordered amongst themselves
 " So that the emerging Proportion on both Sides may be
 " still alike. And it may be answered, that, it may be
 " done after all the Ways and Manners possible; for setting
 " ing the Proportion of A to B, and that of a to b
 " alike, both of them may be expressed by the same Numbers
 " bers after this manner, $A : B :: 9 : 3$, and $a : b :: 9 : 3$
 " And consequently all the Proportions emerging on both
 " Sides, either by *Alternating* the Terms, or by *Inverting*
 " them, or by *Compounding*, or *Dividing*, or *Converting*
 " or *Mixing* them, may be expressed by the very same
 " Numbers; and consequently the same Proportion will
 " always be kept on both Sides. As for Example Suppose
 " $A + B : B :: a + b : b$, because $9 + 3 : 3$, expresseth the
 " same Proportion; which is *Composition*. The same may
 " to be said of all the Ways of changing the Terms
 " Therefore let Beginners observe this one Thing, that
 " Proportions, which are on both Sides the same, be easily
 " changed and ordered in the very same manner. And
 " then there will be no room to question, whether the
 " Proportions which arise on both Sides be alike or not
 " It is indeed a Thing to be wondered at, that no one
 " those who have hitherto compiled Elements of Geometry
 " try, have made use of this most easy Method of stating
 " the Equality of Proportions, for the Illustrating of the
 " Fifth Book about the Doctrine of Proportions. Therefore
 " therefore the primary Ways which Geometry makes use
 " of, in Reasoning concerning like Proportions, as they
 " are digested into this short Table.

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Let it be $A : B :: a : b :: 9 : 3$

Then it will
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Alternating $A : a :: B : b :: 9 : 9 :: 3 : 3$

Inverting $B : A :: b : a :: 3 : 9$

Compounding $A+B : B :: a+b : b :: 9+3(12) : 3$

Dividing $A-B : B :: a-b : b :: 9-3(6) : 3$

Converting $A : A+B :: a : a+b :: 9 : 9+3(12)$

or $A : A-B :: a : a-b :: 9 : 9-3(6)$

Mixing $A+B :: A-B :: a+b : a-b :: 9+3 : 9-3$

Ex æquo $A : B :: a : b$, & $B : C :: b : c$, then $A : C :: a : c$.

$9 : 3 :: 9 : 3$ $3 : 1 :: 3 : 1$, then $9 : 1 :: 9 : 1$.

Ex æquo $A : B :: a : b$, & $B : C :: r : a$, then $A : C :: r : b$.

perturbate $8 : 3 :: 8 : 3$, & $3 : 12 :: 2 : 8$, then $8 : 12 :: 2 : 3$.

Or thus, $8 : 3 :: 16 : 6$, & $3 : 2 :: 24 : 16$, then $8 : 2 :: 24 : 6$:

$a : b :: e + a : a + b$ & $b : e :: a + b : a + e$ then $a : e ::$

$a + b : e + b$.

" He therefore that is expert in these Ways of Reasoning concerning Proportionals, and knows how to bring them into Use upon occasion, will seldom stand in need of the particular Propositions of the Fifth Book. Only two of them, which yet are almost Axioms, may not improperly be inserted and illustrated by Example, in Way of *Appendix*, because of the Frequency of their Use in all Parts of the Mathematicks; which therefore shall be done after the Definitions.

DEFINITIONS.

1. **A**N *Aliquot* Part of Magnitude, is that which being so many times more or less repeated, doth measure, or is just equal to the Magnitude. An *Aliquant* Part is that which doth not measure it.

The Length of one Foot is an *Aliquot* Part of the Length of 10 Feet, because being ten times repeated, it measures it. But the Length of 4 Feet is an *Aliquant* Part of a Line of 10 Feet, because being so many times repeated, to wit, twice, it falls short of it, but being thrice repeated, it exceeds it.

2. One Magnitude is said to be a *Multiple* of another, when the lesser measures the greater, and consequently is

an *Aliquot* Part thereof; or when the greater contains the lesser so many times precisely.

3. Proportion is the mutual Respect, as to Quantity, of two Magnitudes of the same Kind.

Therefore there are in all Proportions two Terms, of which that is called the Antecedent which is first named, or which is named in the Nominative Case; the other the Consequent.

When the Antecedent and the Consequent are equal, it is called Proportion of Equality; when they are unequal, Proportion of Inequality.

4. Rational Proportion, is that which is betwixt commensurable Magnitudes, and may be expressed in Numbers. Irrational Proportion, is that which is betwixt Quantities incommensurable, and cannot be explicated by any Numbers.

Moreover, commensurable Quantities, are those which have some common Measure measureth; Incommensurable, those which cannot be measured by any common Measure.

Fig. 1. l. 5. 5. Two Proportions (that of A to B, and that of C to F) are alike, equal or the same; when the Antecedent of one [A] doth equally, or in the same manner, that is, neither more nor less, contain its Consequent [B] as the Antecedent of the other [C] contains its Consequent [F].

Fig. 2. Or when the Antecedent of the one [A] is so often contained in its Consequent [B] as [C] the Antecedent of the other is in its Consequent [D].

Fig. 3. 6. Two Proportions are unlike, or one is greater than the other, when the Antecedent of one [I] doth more contain its Consequent [L], than the Antecedent of the other [O] doth contain its Consequent [Q;] or when the Antecedent of one is less contained in its Consequent, than the Antecedent of the other in its Consequent.

Fig. 4. 7. Like or *similar* Parts are those which are equally contained in the same manner in their Wholes; so that what sort of Part one is of its Whole, such a Part the other is of its Whole. Which thing indeed is nothing else, but that the Parts bear the same Proportion to their Wholes.

Aliquot Parts are alike, which do equally measure their Wholes, as if each of them be one Third or one Tenth &c. of its Whole.

Fig. 6. 8. Magnitudes [A, B, C, D] are said to be *continued* proportional when the middle Terms [B, C] are taken twice

twice; that is, when they are each of them a Consequent in respect of the foregoing, and an Antecedent, in respect of the following.

We thus pronounce continual Proportions. A is to B, as B to C; and B is to C, as C is to D. And so on.

9. Magnitudes are *discretely* proportional when no Term is twice taken.

Discrete Proportions we thus pronounce: A is to B, as Fig. 1. C to F. When they are more than three proportional Magnitudes if they be said to be proportional, they are always understood to be *discretely* so.

10. When the Magnitudes [A, B, C, D] are continually Fig. 6. proportional, the first [A] is said to have to the third [C] duplicate Proportion of that which it hath to the second [B:] And the first [A] is said to have to the fourth [D] a triplicate Proportion of that which the same first hath to the second [B:] And so forwards.

[" If one triplicate Proportion be equal to another duplicate Proportion, the latter simple Proportion shall be sesquiplicate, or one and a half of the former simple Proportion. Let A, B, C, D, be $\div\div$; and a, b, c, $\div\div$; and let A the first in the former Analogy be unto D the fourth; as [a] the first in the second Analogy is to [c] the third; I say, that [a] is to [b] in a Proportion, which is one and a half of that which A is in to B. For let F be a middle Proportional betwixt B and C: Or, which is the same thing, betwixt A and D. Because of the Equality of the Proportions of A to D, and [a] to [c] and the middle Proportions on both Sides F and [b:] it will be $A : F :: a : b$. But the Proportion of A to F is compounded of the entire Proportion of A to B, and of the Proportion of the same B to F halved; and consequently the Proportion of [a] to [b] which is equal to that of A to F, contains the entire Proportion of A to B, and also the same halved, to wit, the Proportion of B to F. But the whole Proportion, with its half, is a sesquiplicate or sesquialteral Proportion, or that which is one and a half of the other. [a] Therefore is to [b] in a Proportion sesquiplicate of that of A to B. So in Astronomy, since the Cubes of the Distances of the Planets from the Sun bear that Proportion one to another, which the Squares of their periodical Times bear; so that the triplicated Proportion of the Distances, is the

" same

" same with the Duplicate one of the periodical Times
 " it is wont to be said, that the periodical Times are in a
 " sesquiplicate or sesquialteral Proportion of their Distances from the Sun.]

Fig. 1.

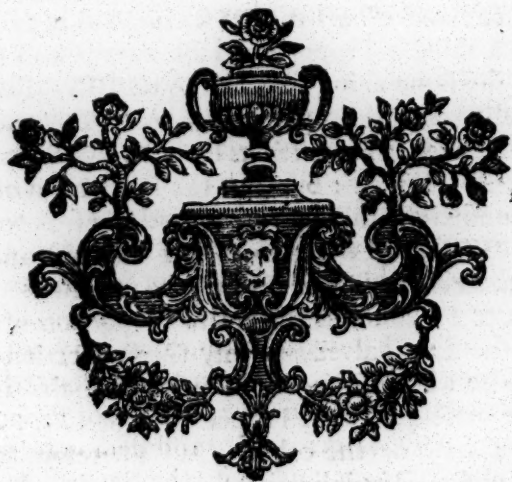
11. Antecedent Magnitudes are said to be Homologous or like to Antecedent, and Consequent to Consequent Magnitudes. As if A is to B, as C to F; A, C, and B F, are homologous Quantities.

XII. If a Set of Pairs of Quantities contain every one the same Proportion, that is the very Proportion also which the Sum of all the Antecedents bears to the Sum of all the Consequents.

$$\begin{array}{r} 20+6+8+18+14=66 \\ \hline 10+3+4+9+7=33 \end{array}$$

XIX. If Parts be as Wholes, the Remainders will be also in the very same Proportion.

If 30 be to 20, as 3 to 2; 27 will be to 18 also as 3 to 20, or as 3 to 2.



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Elements of EUCLID.

B O O K VI.

THE Doctrine of Proportions, which was generally set forth in the Fifth Book, is applied in the Sixth, to plain Figures. And those Things which are delivered in this Book are so necessary to be known, that without them no Man can penetrate into the Secrets of Geometry, and reap the sweet Fruits of the Mathematicks. Each Proposition deserves to have an Encomium annexed; so great is the Utility of all.

" This Sixth Book, as hath been said, begins to apply that excellent Doctrine concerning Geometrical Proportion, which was just before delivered, to divers, and those certainly, most notable Uses; and beginning with Triangles, the most simple of Figures, searches out their Sides and Areas, as they answer to one another in a certain Proportion. Then it defines proportional Lines, and the proportional Augmentations, or Diminutions of Figures; and shews in what manner we may either increase or diminish them according to any Proportion given. It opens likewise the Golden Rule, or Rule of Proportion, the very chief of all Arithmetick; and demonstrates that in a Rectangle Triangle, not only the Square, but also the Pentagon, Hexagon, and in general, every regular Polygon, which is described by the Hypothennuse, is equal to the Squares, Pentagons, Hexagons, or any regular Polygons whatsoever, that are described by the two Sides.

" It

" It also propounds most easy and certain Principles for
 " measuring as well Solids, as Lines and Surfaces
 " which are of very great Use in all Parts of the Mathematicks.

DEFINITIONS.

1. **L**IKE or similar Figures, are those which both have all the Angles equal, each to each other respectively, and the Sides which are opposited to the equal Angles, or which are betwixt them, or which are about the equal Angles, (for they come all to one) Proportional.

Fig. 7. 1. 6. As the Triangles X, Z, will be said to be like, or similar, if the Angle A be equal to the Angle F, and the Angle B equal to the Angle I, and consequently the Angle C equal to the Angle L: And moreover, if A B be to F I, as B C to L I; and B C is to L I, as C A is to L F; and C A to L F as A B to F I; by comparing always the Sides opposite to the equal Angles. In the same manner the Likeness of all right-lined Figures may be explained.

Fig. 29. 2. Reciprocal Figures, are when the Antecedent and Consequent Terms of the Proportions appear on both Sides.

As in the Parallelograms X Z,

If A C be to C B,

As F C is to C L.

The Antecedents here are A C and F C, of which there is one in both Figures; and the Consequents are C B, and C L; of which likewise there is one in each Figure. And therefore the Parallelogram X, Z, are called Reciprocal. Understanding the same of other Figures.

3. The Altitude of a Figure is the Perpendicular let fall from the Top to the Base. This is with *Euclid* the fourth Definition.

Fig. 1. As the Altitude of the Triangle A B C is the Perpendicular A Q, which falls from the Top upon the Base B C either within the Triangle or without, upon the Base produced. Now the Base and Top are assumed at Pleasure.

4. Like Arches of Circles are those which have the same Proportion unto their whole Circumferences.

As if each of them be a Third or Fourth Part, &c. of that Circumference.

PROPOSITION I. Theorem.

Triangles (ABC, DEF) and Parallelograms Fig. 2.
 ($AOPC, DQRF$) which are betwixt the
 same Parallels, or have the same Altitude, have
 the same Proportion betwixt themselves as their
 Bases, (AC, DF).

Upon this Theorem the whole Sixth Book depends, yea,
 whatsoever any where is demonstrated about Figures by
 proportions, whether Plane or Solid.

Let there be taken any Aliquot Part of the Base DF ;
 e.g. DG one Third, and let the right Line GE be drawn:
 the Triangle DEG will likewise be one Third Part of the
 Triangle DEF , as is gathered from 38. 1. 1. Wherefore
 ABC, G , and the Triangle DGE are like Aliquot Parts of
 their Consequents *. Then let there be taken away DG * *Per*
 from the Base AC , as often as it can, as suppose six times, *Def. 7. l. 5.*
 and let the right Lines HB, IB, KB, LB, MB, NB ,
 be drawn. Because the Lines $CH, HI, &c.$ are each of
 them equal to DG , the six Triangles $CBH, HBI, &c.$
 each of them (a) equal to the Triangle DGE . (a) *Per* 38.
 therefore as often as DG is contained in the Antecedent *l. 1.*
 AC , so often is the Triangle DGE contained in the Tri-
 angle ABC . By the same Reasoning it may be shewed,
 that the like Aliquot Parts whatsoever of the Consequents
 the Base DF , and the Triangle DEF are in an equal
 number contained in the Antecedents (the Base AC , and
 the Triangle ABC :) Therefore as the Base AC , is to the
 Base DF ; so is the Triangle ABC , to the Triangle DEF .
 E. D.

But now, because the Parallelograms AP, DR are (b) (b) *Per* 41.
 equable to the Triangles ABC, DEF , they also will be as *l. 1.*
 their Bases.

Corollary.

THE Triangles (ABC, FIL) and the Parallelograms
 which have equal Bases (AC, FL) or the same, have Fig. 3.
 the same Proportion one to another, which their Altitudes ($BO,$
 Q) have.

For

- For let QS , OR , be made equal to the equal Bases (FL , AC); QS , OR will then be equal. Draw SI , RB . If in the Triangles OBR , QIS , you take BO , QI for the Bases, OR , QS , will be their Altitudes, which seeing they are equal, the Triangles OBR , QIS will be betwixt themselves, as their Bases BO , QI .
 (c) *Per* 1. *l.* 1. But because by the Construction, OR is equal to AC and QS equal to FL , the Triangles OBR , QIS , are equal to the Triangles ABC , FIL . Therefore the Triangles ABC , FIL , are also as BO is to QI .
 (d) *Per* 38. *l.* 1. Corollary (2.) " Hence a Trapezium $ABCD$, whose Sides AD and BC are parallel, may be divided into any equal Parts whatsoever. For let CE be made equal to AD . Because of the Equality of the Angles vertically opposite (*e*) AFD , $EF C$, and of the alternate Angles (*f*) DAF , FEC , and ADF , ECF , and the Equality of the Bases AD , CE , by Construction, the Triangles ADF , FCE (*g*) are equal; and therefore the Triangle ABE is equal to the Trapezium $ABCD$. Therefore the Base BE being divided into any equal Parts whatsoever; as for instance, three, BG , GR , RE the Triangles ABG , AGR , ARE , shall each of them be one Third Part of the Trapezium. *Q. E. I.*

PROP. II. Theorem.

Fig. 4. **I**f to one Side of a Triangle (as BC) there be drawn (FL) a Parallel, this cuts the Sides proportionally, that is, (AF) will be to (FB) as (AL) to (LC):

And if the right Line (FL) cuts the Sides (BA , CA) proportionally, it will be parallel to the other Side (BC).

- Part I. Let BL , CF be drawn, because FL is supposed parallel to BC , the Triangles FBL , LCF having the same Base are (*a*) equal. Therefore the Triangle X has the same Proportion to both; now the Triangle X is to the Triangle FBL , as the same Triangle X is to the Triangle LCF . But the Triangle X is to the Triangle FBL as AF is to FB ; and the Triangle X is to that LCF as AL (*c*) is to LC . Therefore also AF is to FB , as AL to LC . *Q. E. D.*

Part II. As AF is to FB , so is the (d) Triangle X to (d) By the Triangle $FB L$: And as AL is to LC , so is the same *foregoing*. Triangle X to the Triangle $LC F$. Now AF is supposed to be to FB , as AL is to LC . Therefore the Triangle X is to the Triangle $FB L$, as the same X is to $LC F$. Therefore the Triangles $FB L$, $LC F$ are equal. Therefore seeing they have a common Base FL , the Lines AB , BC , are (e) parallel. Q. E. D.

(e) Per 39.
l. 1.

Corollary.

unto (BC) one Side of a Triangle there be drawn more *Fig. 5.* Parallels (IO, FL) all the Segments of the Sides will be proportional.

Let FQ be drawn parallel to AC . The right Lines FS , Q , are equal (f) to LO , OC . But BI is to FI , as (f) Per 34. SO to SF (a) . Therefore BI is also to IF , as CO to L .

(a) Per 2.
l. 6.

PROP. III. Theorem.

A right Line (BF) which bisects an Angle *Fig. 6.* of a Triangle, doth also cut the Base (AC) , the Segments of the Base (AF, FC) will have the same Proportion betwixt themselves as the Sides AB , BC have.

And if the Parts of the Base (AF, FC) have the same Proportion betwixt themselves, as the Sides (AB, CB) the Line (BF) which cuts the Base, bisects the opposite Angle (ABC) .

Part I. Draw from CB until BL be equal to BA ; join AL . Because in the Triangle Z , the Sides LB , BA , are equal, the Angles also (b) L and O are equal. (b) Per 5. Therefore the external Angle ABC is equal to the (c) L , O , the Angle I , which by the (c) Per 32. Hypothesis is half ABC , will be equal to the Angle L . (d) 1 . Therefore AL , FB (d) are parallel. Therefore in the (d) Per 29. Triangle ACL , AF is to FC (e) as LB (that is, AB) is to LC . Q. E. D.

(e) Per 2.

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Part 6.

Part II. Protract CB again, until BL be equal to BA.

- Because AF is supposed to be to FC, as AB (that is, BL) is to BC; AL, FB (*a*) are parallel. Therefore the internal Angle I is equal to the internal one L (*b*); and alternate Q equal to the alternate O. But because AB, BL, are equal, the Angles L and O (*c*) are equal. Therefore I and Q are also equal. Therefore ABC is bisected by AL.
- Q. E. D.*

PROP. IV. Theorem.

Triangles which are Equiangular to one another, are like or similar, that is, have the

- Sides also (*a*) that are opposite to the equal Angles proportional.
- Def. 1. l. 6.*

Fig. 7. In the Triangles XZ, let the Angle A be equal to the Angle F, and the Angle C to the Angle L, and the Side AB to the Side FI; I say, that AB is to FI, as AC is to FL; and AC is to FL, as CB is to LI; and CB is to LI, as BA is to FI.

Fig. 7, 8. *Demonst.* If the Angle F be laid upon its equal A, Sides FI, FL, will fall upon the Sides AB, AC. And because the external Angle AIL is by the Hypothesis equal to the internal B (*b*), therefore (*c*) IL, BC, are parallel.

(b) Fig. 8. Therefore BI is to IA (*d*) as CL to LA. Therefore alone.

(c) Per 29. compounding, BA is to IF, as CA to LF. And if Angle L be laid upon the Angle C, it will be shewed in the same manner, that AC is to FL, as BC is to IL; and the Angle I be laid upon the Angle B, it will be shewed the same manner, that BC is to IL as AB to FI. Proposition therefore is proved.

(d) Per 2. *l. 6.*

Corollaries.

Fig. 8. **1.** IF in a Triangle a Line LI be drawn parallel to the Side BC, the Triangle LFI will be like to the Whole Triangle BCF; and consequently CF will be to LF, as BC to LI.

For because LI, BC, are parallel, the external Angles FIL, FLI will (*per 27. l. 1.*) be equal to the internal ones B and C: But F is common to both Triangles. Therefore the Triangles are similar.

they are equiangular. Therefore the Sides CF , LF opposite to the equal Angles B and FIL (a) are proportionally (a) *By the* to the Sides BC , LI , which are opposed to the corresponding Angle F .

If in a Triangle a right Line BF , drawn from the opposite Angle B , doth cut the Parallels AC , LO , it cuts them proportionally.

or by *Corollary 1*. AF is to LI , as FB is to IB ; and also is to IO , as FB is to IB . Therefore AF is to FC to IO . Therefore by changing, AF is to LI to IO .

From *Corollary 1*. " We learn to find the Height of a Tower, or any elevated Point, by only the Shadow of a Staff. Fix the Staff FL perpendicularly upon the ground in that Place where the Ray of the Sun XBA , terminates the Shadow of the Tower BZ , may also pass through L . There will be in the Triangle AZB , the Line FL , parallel to the Height of the Tower ZB . Hence as AF , the Distance of the Staff from the Point of the Shadow, is to FL , the Length of the Staff; so is AZ , the Distance of the Tower, from the Point of the Shadow, to ZB , the Height of the Tower. And because the three first Terms are easily had by measuring the fourth also, *i. e.* the Height of the Tower is had.

Q. E. I.

From this also incomparably useful Proposition, we may deduce that famous Theorem of *Ptolemy*; to wit, that in every Quadrilateral inscribed in a Circle, the Rectangle of the Diagonals $AC \times BD$ is equal to the Sum of the Rectangles of the opposite Sides, $AB \times CD$ and $AD \times BC$. For let the Angle BAE be made equal to Angle CAD . Because the Angles BAE , CAD , are equal by Construction, the Angles ABE , ACD , standing upon the same Arch AD , are equal; therefore the Triangles BAE , CAD , are alike. And $AC : AE :: AB : BE$; and consequently the Rectangle of the Extremes $AC \times BE$ is equal to the Rectangle of the Means $CD \times AB$. In like manner, because the Angle ADB is equal to the Angle BAC by Construction, the Angles ADE , ACB , as standing upon the same Arch AB , are equal: The Triangles ADE , ACB , will be like; and $AD : DE :: AC : CB$. And therefore the Rectangle of the Extremes $AD \times CB$ is equal to the Rectangle of the Means $AE \times BE$.

And if we may deduce that famous Theorem of *Ptolemy*; to wit, that in every Quadrilateral inscribed in a Circle, the Rectangle of the Diagonals $AC \times BD$ is equal to the Sum of the Rectangles of the opposite Sides, $AB \times CD$ and $AD \times BC$. For let the Angle BAE be made equal to Angle CAD . Because the Angles BAE , CAD , are equal by Construction, the Angles ABE , ACD , standing upon the same Arch AD , are equal; therefore the Triangles BAE , CAD , are alike. And $AC : AE :: AB : BE$; and consequently the Rectangle of the Extremes $AC \times BE$ is equal to the Rectangle of the Means $CD \times AB$. In like manner, because the Angle ADB is equal to the Angle BAC by Construction, the Angles ADE , ACB , as standing upon the same Arch AB , are equal: The Triangles ADE , ACB , will be like; and $AD : DE :: AC : CB$. And therefore the Rectangle of the Extremes $AD \times CB$ is equal to the Rectangle of the Means $AE \times BE$.

- " equal to the Rectangle of the Means $D E \times A C$. But
 (a) *Per* 1. " Rectangles $A C \times B E$, and $A C \times D E$, are equal (a) to
 l. 1. " Rectangle $A C \times B D$. Therefore the Rectangles $A B \times D$
 " and $A D \times B C$, which are made by the opposite Sides
 " are equal to the Rectangle $A C \times B D$, which is made
 " the Diagonals. *Q. E. D.*

PROP. V. Theorem.

Fig. 10. **I***F two Triangles have all their Sides proportional each to each, they shall also be mutually Equiangular.*

That is, if $A B$ be to $R F$, as $A C$ to $R Q$; and as $A B$ is to $R Q$, so is $C B$ to $Q F$; and as $C B$ is to $Q F$, so is $A B$ to $R F$; I say, that the Angles opposite to the antecedents, are equal to the Angles opposite to the consequents; to wit, C to I , and B to F , and A to O .

Ang.	Antec.	Conseq.	Ang.
C	$A B$	$R F$	I
B	$A C$	$R Q$	F
A	$C B$	$Q F$	O

- Make X and Z equal to A and C ; and let the
 (a) *Per* meet in N . The Angles B and N will (a) be also
Corol. 9. Because therefore the Triangles P, T , are Equiangular
p. 32. l. 1. $A B$ (by the foregoing) will be to $R N$, as $A C$ to $R Q$.
 But by the Hypothesis, $A B$ is to $R F$, as $A C$ to $R Q$.
 Therefore $A B$ is to $R F$, as the same $A B$ is to $R N$.
 Therefore $R N, R F$, are equal. In the like manner
 might shew that $Q N$ and $Q F$ are equal. Therefore
 the Triangles T, S , are equilateral to each other.
 Therefore the Angles, I, F, O , are equal (*per* 8. l. 1.).
 Angles Z, N, X , that is, by the Construction to the
 angles, C, B, A . *Q. E. D.*

PROP. VI. Problem.

Fig. 10. **I***F two Triangles (P, S) have an Angle equal to one Angle (O); and the Sides ($AC, R F, R Q$) which contain the equal Angle, be proportional; the Triangles will be like.*

Let X and Z be made equal to the Angles A C, and the
 des meet together in N. Therefore the Angles B and N
 will † be also equal. Then it may be shewed, as in the (†) *Per*
 regoing, that R F, R N, are equal. But R Q is common *Corol. 9.*
 both Triangles S, T. The Angles also O and X are *p. 32. l. 1.*
 equal, because they are equal to the same A, the one X by
 the Construction, and O by the Hypothesis. Therefore (c) (c) *Per 4.*
 and F are likewise equal to Z and N. Therefore the *l. 1.*
 triangle S is Equiangular to the Triangle T; that is, by
 the Construction, to the Triangle P. Therefore S, P are
 equal (per 4. l. 6.) Q. E. D.

PROP. VII.

S scarce of any Use.

PROP. VIII. Theorem.

*In a Rectangle Triangle, the Perpendicular Fig. 11.
 (BC) let down from the right Angle to the
 Hypotenuse, cuts the Triangle into Parts like to the
 whole, and betwixt themselves.*

In the Triangles A B F and L, the Angle F is common,
 and the Angles A B F and X are, by the Hypothesis right
 angles, and consequently equal. Therefore the other Angles
 and O are (b) also equal. Therefore * the Triangles (b) *Per*
 A B F and L are like: In the same manner the Triangles *Corol. 9.*
 A B F and R may be shewed to be equal, and the Angle *Prop. 32.*
 equal to the Angle F. From which it is now manifest *l. 1.*
 that R and L also are like, seeing the Angles I and F, O * *Per 4.*
 A, U and X, are equal. Q. E. D. *l. 6.*

Corollaries.

FIRST, B C is a mean Proportional betwixt A C
 and C F.

For seeing there be in the Triangle R and L,
 equal Angles, I, F } { equal Angles, A,
 Sides opposed, A C, C B } { Sides opposed, C B, C

(a) *Per* 4. It is manifest (a) that $AC : CB : CF \div$
 l. 6. 2. BF is a mean Proportional betwixt AF, and C
 Likewise AB, a mean betwixt FA, and CA.

For in the Triangle ABF and L.

equal Angles, ABF, X } { equal Angles, A,
 Sides opposed, AF, BF } { Sides opposed, BF, C

(b) *By the* Therefore AF (b) : BF :: BF : CF. Likewise beca
same. in Triangle ABF and in R there be

equal Angles, ABF, V } { equal Angles, F
 Sides opposed, AF, AB } { Sides opposed, AB, A

It will be again $AF : AB : AC \div$

Fig. 11. 3. " Hence we learn to measure an inaccessible Line
 " one Term whereof is accessible. Let the inaccessible
 " Line be CF. Let there be raised from the Point C
 " Perpendicular CB: And to any Point of this Perpend
 " ular, as B, let there be applied a Square, or any rig
 " Angle ABF; so that in looking along the Line B
 " the Point F, and along the Side BA, the Point A m
 " be observed. Let the accessible Line AC be measure
 " and from the following Analogy the inaccessible CF w
 " be made known. $AC : CB :: CB : CF$. Let the Squa
 " then of the Line CB be divided by the Line AC, a
 " the Quotient (c) will give the sought Line CF. Q. E.

(c) *Per.*
Corol. 3.
p. 17. l. 6.

PROP. IX. Problem.

Fig. 12. **TO divide a given Line (AB) according to
 given Proportion FI to IL).**

Let the infinite Line AZ be drawn. From which ta
 A Q, QR, equal to FI, IL. From R draw RB. P
 rallel to this, draw QC from Q. I say, the Thing
 done.

It is manifest from *Proposition 2. L. 6.*

PROP. X. Problem.

TO divide a given Line, as (AB) in like man-*Fig. 13.*
ner as another given one (AI) hath been di-
vided (in F, G .)

Let the right Line IB join the Extremities of the two
Lines. Draw Parallels to this from the Points, F, C ,
which may meet the right Line, that is to be cut, AB in
and Q . I say the Thing is done.

This is manifest from the *Corollary* of *Proposition 2. l. 6*.

[Or thus, if the cut Line IA be greater than that *Fig. 53.*
which is to be cut, BQ , let three Circles touching one
another, be described with the Diameter IF, IC, IA ;
and let the Subtense BQ be fitted from the Point I to
the Circumference of the greatest Circle: the two lesser
Circles will cut the Line BQ in the Points L, P , in the
Proportion * of the Sections of the Diameter IA . If* *Per Co-*
the Line IA be cut into four Parts, four Circles are to *rol. 4. p. 31.*
be drawn; if into five, then five Circles; and so infinite-
ly.]

Scholium.

FROM this Proposition we learn to cut a right Line given *Fig. 13.*
into any equal Parts whatsoever. Let an infinite right
Line make any Angle with the right Line to be cut, AB ;
from which take, with a Pair of Compasses, so many equal
Parts, AC, CF, FI , as you would divide AB into.
Draw the right Line IB , and the Parallels to it, FL, CQ .
I say the Thing is done.

We may do the same Thing otherwise, and more easily *Fig. 14.*
after *Maurolycus*, in the manner following. Let AB be to
be trisected or divided into three equal Parts. Draw the
infinite Line IX parallel to AB , above or below it. From
 X , if it be below AB , take with a Pair of Compasses
three equal Parts, IQ, QR, RS , which together may be
greater than AB ; but lesser, if IX is above. Through I
and A , as likewise through S and B , draw right Lines,
which may meet together in C . From C to Q and R draw
right Lines: These will trisect the given Line AB . The
demonstration appears from *Corollary 2. Prop. 4.*

Again,

Fig. 15. Again, with *Maurolicus*, we may otherwise obtain the same thing, to wit, thus: Let AB be to be quadrisectioned. Draw the infinite Line AX , and BZ also an infinite Line parallel to it. From these take with the Compasses equal Parts AL, LO, OQ , and BV, VS, SR , in each several Part by one than are required in AB ; then let there be drawn the right Lines, LR, OS, QV . These will quadrisection the given AB .

For because by Construction, the Lines LO, RS , parallel and equal, are joined by LR and OS , these also will be parallel. In the like manner OS and QV are parallel. Therefore seeing AQ is cut into three equal Parts, AI will also *(b)* be cut into so many equal Parts. Likewise BC will be cut into three equal Parts. Therefore the whole AB will be cut into four equal Parts.

These two Ways of Practice are easier than *Euclid's*, because fewer Parallels are to be drawn.

PROP. XI. Problem.

Fig. 16. *TO find a third Proportional to two right Lines given (AB, BC).*

Draw the right Line AC . From BA produced, take AF , equal to BC . Through F , draw the infinite Line FX , parallel to AC , which Infinite, let BC produced meet in L . I say that AB is to BC , as BC to CL .

(b) Per 2. For $AB : AF$ *(b)* :: $BC : CL$. But AF *(c)* is equal to BC . Therefore $AB : BC$:: CL ; and so CL is the third Proportional sought.

Construction.

Otherwise.

Fig. 17. **L**ET AB and BC be set at a right Angle. Join AC . From C draw CX perpendicular to AC infinite. From A draw AX parallel to BC , which CX , let AB produced, meet, in L . I say, $AB : BC$:: $BC : BL$. It is manifest from *Corollary 1.*

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Scholium.

A Given Proportion may not only be continued in three, but also in infinite Terms, and the whole Sum of the infinite proportional Terms be exhibited. *Gregory of Saint Vincent* hath very handsomly prosecuted this Matter, and the whole Business of Geometrical Progression in the whole Second Book of his Work. We, for the sake of the Studios, will here present succinctly the Construction and Demonstration of the Thing proposed.

Problem.

LET a Proportion of the greater Inequality be given, as *Fig. 19.*
 AB to BC . It is required to continue this through infinite Terms, and to present the Sum of them all.

Let the Perpendiculars AL , BO , be erected, and taken equal to the given Lines AB , BC , and through L , O , let a right Line be drawn, meeting with ABC produced in Z .
 I say, 1. If from C you erect the Perpendicular CQ ; CQ shall be a third Proportional. Transfer QC into CE , and from E erect ER ; this shall be a fourth Proportional. Transfer ER into EF , and erect FS ; this shall be a fifth Proportional: And so the Proportion of AB to BC , that is, of AL to BO , will be continued through the Terms AL , BO , CQ , ER , FS , &c. or AB , BC , CE , EF , &c. infinitely, because every Term (as FS) may be taken away from the remaining one FZ ; for seeing LA (that is, AB) is less than AZ ; FS also (a) must ever be less (a) *Per*
 than FZ . *Corol. 1.*

Prop. 4.

l. 6.

I say, 2. AZ is equal to the whole Sum of the infinite Proportionals.

Part I. [“ It being supposed as before, $AZ : BZ :: AB : BC$; it will be by alternating $AZ : AB :: BZ : BC$. And by dividing, $AZ - AB : AB :: BZ - BC : BC$; that is, $BZ : AB :: CZ : BC$. Therefore by inverting $AB : BZ :: BC : CZ$. And by compounding $AB + BZ : BZ :: BC + CZ : CZ$; that is, $AZ : BZ :: BC : CZ$ “ :: BZ

" :: BZ : CZ." But as AZ is to BZ, so is LA to OB; and as BZ is to CZ, so is OB to QC. Therefore also LA is to OB, as OB is to QC. In the same manner I might shew that OB is to QC, as QC to RE; and so forwards infinitely.

Part II. The whole Sum of the infinite Terms is neither less than AZ, nor greater; therefore it is equal. It is not greater, because seeing we have shewed above, that QC is lesser than CZ, and RE than EZ, and SF than FZ, and so on infinitely, all the Terms QC, RE, SF, &c. may be infinitely set one by another in the right Line AZ; so that the Point Z shall never be reached. Again the said Sum will not be less, because I have above shewed AZ, BZ, CZ, to be continually proportional; and in the same manner, the same thing is shewed of the rest, EZ, FZ, &c. Seeing therefore by transferring the Proportionals, QC, ER, FS, &c. into CE, EF, FI, the Remainders EZ, FZ, IZ, &c. are always continually proportional, as we have already shewed; we shall at the last come unto a Remainder less than any given one; and therefore the Sum of the Proportionals shall exceed every Quantity that is less than AZ; from whence itself cannot be less than AZ. Seeing therefore it is neither greater nor less than AZ, it shall be equal to it. Q. E. D.

Theorem.

THE Difference of the first Terms, the first Term, and the whole Sum of the infinite Proportionals, are continually proportional.

Fig. 19.

(a) *Per*

Corol. 1.

Prop. 4.

l. 6.

Fig. 20.

In the upper Figure let OX be drawn parallel to AZ. Therefore LX shall be the Difference of the first Term AL or AB, and of the second BO, or BC. Because XO is parallel to AZ; LX shall be to XO, as (a) LA is to AZ. But XO is AB, and LA likewise is AB. Therefore the Difference LX is to the first Term AB, as AB the first Term is to AZ, the whole Sum. Q. E. D.

The same thing may be demonstrated universally and very briefly in every kind of Quantity; thus, Let there be any continual Proportionals whatsoever (as well Numbers, as other Quantities) AZ, BZ, CZ, &c. and let them all be transferred upon the first AZ. Therefore AB, BC, CB

E B

$F, FI, \&c.$ will be the Differences of the Proportionals; which, together with the last Quantity IZ , are equal to the first AZ . Now, because if Proportionals be continued infinitely, the last Quantity vanisheth away, it is manifest that the Differences of the infinite Proportionals are equal to the first AZ . Then, because AZ is to BZ , BZ is to CZ , and so on. By dividing, AB will be to Z , as BC to CZ ; and by converting, as AB , the first Difference, is to AZ , the first Quantity; so BC the second Difference, is to BZ , the second Quantity, and so onwards. Therefore as AB , the first Difference, is to AZ the first Quantity. So all the Differences (that is, as have already shewed, the first Quantity AZ) are to all the Quantities, that is, to the whole Sum of the infinite quantities. *Q. E. D.*

PROP. XII. Problem.

THREE right Lines being given (AB, BC, AF) to find a fourth Proportional.

Let the two right Lines be disposed, as the Figure *Fig. 21.* shows, and draw the right Line BF , to which let the infinite right Line CZ be made parallel. Let AF produced L , meet CZ .

I say, AB is to BC , as AF to FL , as is manifest from *Proposition 2.* of this Book. Therefore FL is the fourth proportional sought.

Scholium.

OUR Countryman *Bettin*, in his Treasury of Mathematical Philosophy, doth handsomely from 35. l. 3. and of this, which depends not upon the present Proposition, find out a fourth Proportional, three being given, and a third, two being given, after this manner.

If three right Lines be given, let the second CB , and *Fig. 22.* the third BD , be joined right to one another, so as to make a right Line, and let the first BA touch them in the Point in what Angle you will. Through the Points C, A, D , describe

(a) *Per* 5. describe a Circle (*a*), which let A B, the first Line, meet in the Point Z. B Z is a fourth Proportional.

(b) *Per* 35. For seeing the Rectangles A B Z, C B D are (*b*) equal
l. 3. A B will be to B C, as B D to B Z, by the 14th of this Book, which, as was said, depends upon this.

Fig. 23. If there be given two right Lines, A B, B C; let B D equal to B C, be joined to B C, so as to make one straight Line. Then let the first A B touch B C in B in any Angle. Then the rest is as before, and B Z will be the third Proportional sought.

The Demonstration is the same; for seeing the Rectangles A B Z, C B D, are equal, A B will be to B C, as B D (that is, B C) is to B Z.

P R O P. XIII. Problem.

Fig. 24. **TWO** right Lines given (A C, C B) to find mean Proportional.

Let the whole compound Line A B be bisected in O, and from the Center O a Circle be described through A and B; from C erect a Perpendicular C F, meeting the Circumference in F.

I say, A C is to C F, as C F is to C B.

(c) *Per* 31. For let A F, B F be drawn; the Triangle (*c*) A F B is right-angled, and from the right Angle there is drawn the Perpendicular F C to the Base. Therefore A C is to C F, as C F is to C B.

(d) *Per* as (*d*) C F is to C B.

Corol. 1.

p. 8. *l.* 6.

Corollary:

HENCE it is manifest, that if from any Point of the Circumference (as F) there be drawn a Perpendicular (F C) to the Diameter, this Perpendicular is a mean Proportional betwixt the Segments of the Diameter (A C B)

Scholium

Lib. VI.

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Scholium.

THIS Place requires, that we should say something briefly concerning the finding out of the two mean Proportionals betwixt the two given Lines. All the Geometricians of Greece, at Plato's Suggestion, set themselves with all their Might to the Solution of this Problem. Divers most subtil Ways of Practice are recited by Eutocius in his Commentary on Archimedes; as those of Plato, Architas the Tarentine, Menæchmus, Eratosthenes, Philo Byzantius, Hero, Apollonius of Perga, Nicomedes, Diocles, Sporus, Pappus; to whom the latter Times have added Verner, Gregory of Saint Vincent, Renatus, Cartesius. Out of all these we shall select Three more easy than the rest.

Plato's Method.

IT is required to find out two Means betwixt the given Fig. 29. Lines A B, B C.

Let A B, B C be set in a right Angle, and be produced infinitely towards X and Z. Then let two Squares (so our Claudius Richards hath it; for Plato himself made use of one Square only, but which had inserted into its Side * D E, * See a Rule moveable along D E, let two Squares, I say, be Fig. 26. taken, and the Angle D of one Square be applied to the right Line B X, in such certain wise, that one Side may also pass through A; and to the Point E, in which the other Side cuts the right Line B Z, let a second Square be applied, which will pass through C. I say, that B D, B E, are two Means betwixt the given Lines A B, B C; that is, as A B is to B D, so is B D to B E, and B E to B C.

The Demonstration is manifest from Corollary 1. Prop. 8. L. 6. for A D E is a right-angled Triangle, and from the right Angle to the Base there falls the Perpendicular D B. Therefore by the said Corollary, as A B is to B D, so is B D to B E; and for the same Cause, as B D to B E, so is B E to B C. Therefore betwixt the given right Lines A B, B C, there are found two mean Proportionals B D, B E. Which was the Thing to be done. This Manner of solving the Problem is the easiest of all to be understood.

The Method of Philo the Byzantine.

- Fig. 27.** **L**ET the two given right Lines AB, BC , be set together at a right Angle; then let the Rectangle $ABCD$ be perfected, and let DA, DC be produced infinitely, and let the Diameters BD, AC be drawn, cutting each other in E . From the Center E , through B , let a Circle be drawn, which because ABC is a right Angle (*a*) will pass through A and C . Then let a Rule be applied to the Point B , so that the intercepted right Lines BG, OF , may be equal. I say, that AF, GC , are two mean Proportionals betwixt the given AB, BC ; that is, as AB is to AF , so is AF to GC , and GC to CB .
- (a) *Per 31.* which because ABC is a right Angle (*a*) will pass through A and C . Then let a Rule be applied to the Point B , so that the intercepted right Lines BG, OF , may be equal. I say, that AF, GC , are two mean Proportionals betwixt the given AB, BC ; that is, as AB is to AF , so is AF to GC , and GC to CB .
- (b) *By the Demonstr.* Because GB, OF (*b*) are equal OG, BF , will be also equal. Therefore the Rectangles OGB, BFO , that is, (*c*) the Rectangles DGC, DFA , are equal.
- (c) *Per* Therefore as GD is to DF , so (*d*) reciprocally AF is to GC , but GD is to DF (*e*) as BA to AF . Therefore as BA is to AF , so AF is to GC . Again, because I have already shewed that AF is to GC , as BA is to AF ; and since BA is to AF , as GD is to DF ; that is, GC is to CB , AF will also be to GC as GC is to CB . Therefore all four, BA, AF, GC, CB , are continually proportional; and therefore betwixt the given Lines AB, BC , two Means have been found. *Q. E. I.*
- (d) *Per 14.* already shewed that AF is to GC , as BA is to AF ; and since BA is to AF , as GD is to DF ; that is, GC is to CB , AF will also be to GC as GC is to CB . Therefore all four, BA, AF, GC, CB , are continually proportional; and therefore betwixt the given Lines AB, BC , two Means have been found. *Q. E. I.*
- (e) *Per* CB, AF will also be to GC as GC is to CB . Therefore all four, BA, AF, GC, CB , are continually proportional; and therefore betwixt the given Lines AB, BC , two Means have been found. *Q. E. I.*
- Corol. 1.* all four, BA, AF, GC, CB , are continually proportional; and therefore betwixt the given Lines AB, BC , two Means have been found. *Q. E. I.*
- p. 36. l. 3.* BA is to AF , so AF is to GC . Again, because I have already shewed that AF is to GC , as BA is to AF ; and since BA is to AF , as GD is to DF ; that is, GC is to CB , AF will also be to GC as GC is to CB . Therefore all four, BA, AF, GC, CB , are continually proportional; and therefore betwixt the given Lines AB, BC , two Means have been found. *Q. E. I.*
- p. 4. l. 6.* BA, AF, GC, CB , are continually proportional; and therefore betwixt the given Lines AB, BC , two Means have been found. *Q. E. I.*

These two Methods of Solution, although they be ingenious and easy enough; yet because a due application of a Square and Rule is not made but by trying, they are not Geometrical.

The Method of Cartes.

- Fig. 28.** **L**ET an Instrument of such sort be provided, that two Rules may be opened and shut about Y . Let there be inserted into these divers Squares connected together betwixt themselves in the Points B, C, D, E, F, G , in such sort, that in the mean while that the Rules YX and YZ are opened, the Square BC may impel the Square CD in the Rule YZ , and the Square CD may impel the Square DE in the Rule YX , and the Square DE may impel FE , and

E, F

impel or force forward FG , and so on: But so that the Rules XY and YZ are shut, all the Points B , D , E , F , G , tend to fall upon one and the same Point

By this Instrument not only two, but also four and yea, as many Means as you will, betwixt two given right Lines, may be found. Which thing can be obtained either by the Sections of a Cone, nor by any Methods and out by the abovesaid Authors.

For two Means, three Squares are required; for four means, five Squares, and so on.

Let the lesser of the given right Lines be transferred on the Rule YX , and let it be YB ; the greater upon the Rule YZ , and let it be YE . Let the first Square be applied to the Point B , and be fixed there, and let the Rules be opened, until the Side of the third Square passeth through E . I say, that YC , YD , are two Means betwixt the given YB , YE , that is that YB is to YC , as YC is to YD , and YD to YE .

The Demonstration appears out of *Corollary 2. p. 8. l. 6.* or from the Nature of the Instrument, in the Triangle CD , the Angle at C is a right one, and from it CB falls perpendicular upon the Base YD . Therefore by the said Corollary, as YB is to YC , so is YC to YD . Again, because in the Triangle YDE , the Angle at D is a right one, and from it there falls the Perpendicular DC upon the Base YE , as YC is to YD , so is YD to YE . Therefore YB , YC , YD , YE , are four continual Proportionals. betwixt the given Line therefore YB , YE , there have been found two mean Proportionals, YC , YD . *Q. E. I.*

If betwixt the given ones YB , YG , there be required four Means, open the Rules, until the Side of the fifth Rule FG , passeth through G . There will be YC , YD , YE , YF , four Means betwixt YB , YG . The Demonstration is manifest from the said Corollary.

This Way, although the Instrument is more operose than *Plato's*, is in very Deed an excellent one; both because it doth nothing by bare Trial, and because it extends itself unto four and six, and as many Means as you will.

The *Deliacal* Problem, to wit, the Duplication of the Cube, is performed by two Means, and all the Bodies (a) *See thatsoever* are increased or diminished in a given Proportion (a) by the same; like as the same thing is performed in *l. 12.*

plain

(a) *Cor. 3.* plain Figures (*a*) by one Mean. *Hippocrates* first opened p. 20. l. 6. this way, which, as the Singular and only one, all Geometricians that have followed him, have embraced.

PROP. XIV. Theorem.

Fig. 29, 30. **E**qual Parallelograms (*X, Z*) which have one Angle (*C*) equal to one (*O*); have their Sides also, which are about the equal Angles, reciprocal; that is, (*AC* is to *CB*, as *FO* is to *OL*). And if they have the Sides thus reciprocal the Parallelograms are equal.

Part I. Let *IL* and *SB*, being produced, meet together in *Q*. The Parallelogram *X* is to the Parallelogram *R*, as *AC* is to *CB* (*b*); and *Z* is to *R* (*c*), as *FO* to *OL*. Because, by the Hypothesis, *X* and *Z* are equal, *X* is to *R* as *Z* is to *R*. Therefore also *AC* is to *CB*, as *FO* to *OL*. *Q. E. D.*

(b) *Per 1.* *AC* is to *CB* (*b*); and *Z* is to *R* (*c*), as *FO* to *OL*. Because, by the Hypothesis, *X* and *Z* are equal, *X* is to *R* as *Z* is to *R*. Therefore also *AC* is to *CB*, as *FO* to *OL*. *Q. E. D.*

(c) *By the same.* Part II. As *AC* is to *CB*, so *X* is to *R* (*d*): And *FO* is to *OL*, so is *Z* to *R*. But already by the Hypothesis, *AC* is to *CB*, as *FO* to *OL*. Therefore *X* is to *R*, as *Z* is to *Z*. Therefore *X* and *Z* are equal. *Q. E. D.*

(d) *By the same.*

[Corollary. "On this depends the Demonstration of the inverse Rule of Proportion. For in it there is always some Rectangle given, as *X*; and one Side of another equal Rectangle, as *CB*; and the other Side is sought. As therefore *AC*, the first Side of the given Rectangle is to *CB*, the given Side of the other Rectangle; reciprocally, *FC*, the sought Side, is to *CL*, the second Side of the given Rectangle. The Rectangle therefore *CB* $\times$ *FC*, is equal to the Rectangle *AC* $\times$ *CL*: And the latter Rectangle given being divided by the given Side *CB*, the Quotient will give the sought Side *FC*. *Q. E. I.*

PROP. XV. Theorem.

Fig. 31, 32. **E**qual Triangles (*ACL, FCB*) which have one Angle (*C*) equal to one (*O*) have also the Sides

Propositions about the equal Angles reciprocal (that is, AC is to CB , as FO to OL .)

And if they have their Sides thus reciprocal, the Triangles are equal.

Let the right Line LB be drawn; the rest of the Demonstration is the same as that of the foregoing.

Corollary.

As well Parallelograms as Triangles, which have their Bases and Altitudes reciprocal, are equal: And so conversely.

It is manifest from the two foregoing Propositions.

PROP. XVI. Theorem.

If four right Lines ($AB, FI; IL, BC$) be proportional, (that is, if AB be to FI as IL to BC) the Rectangle (X) under the Extremes AB, BC is equal to the Rectangle (Z) under the Means, (FI, IL). Fig. 33.

And if a Rectangle under the Extremes be equal to a Rectangle under the Means, those four right Lines will be proportional.

Part I. In the Rectangles X and Z , about the right, and therefore equal Angles, B, I , by the Hypothesis. AB is to FI as reciprocally, IL to BC . Therefore X and Z (a) (a) Per 14. are equal. Q. E. D. l. 6.

Part II. Because X and Z are now supposed equal; therefore (b) about the equal Angles B and I , AB is to FI , (b) By the same reciprocally, IL to BC . Q. E. D. same.

Corollary (1.) "Hence it is easy to apply the given Rectangle Z (c) to the given right Line AB ; to wit, (c) Per 12: by making $AB : FI :: IL : BC$. For BC is the l. 6. Rectangle Z applied to the given right Line AB .]

Corollary (2.) "Upon this Proposition depends the Demonstration of the direct Rule of Proportion. For in

" it there is always given some Rectangle, as CL : and
 " another like Rectangle is sought, one Side whereof is al
 " so given. It will therefore be, as BC , the first Side o
 " the Rectangle given, is to EO , the Side of the Rectan
 " gle sought; so directly CE , the second Side of the Rect
 " angle given, is to OA , the other sought Side. There
 " fore the Rectangle $CE \times OE$ is equal to the Rectangle
 " $BC \times OA$. And the Rectangle $CE \times EO$ being divide
 " by BC the Quotient, will give AO , the other Side
 " which was sought. *Q. E. I.*

PROP. XVII. Theorem.

Fig. 34. **I**F the three right Lines (AB, FL, BC) be
 proportional, the Rectangle under the Extremes (AB, BC) shall be equal to the Square of the Mean (FL).

And if the Rectangle under the Extremes be equal to the Square of the Mean, those three right Lines are proportional.

Part I. Let O be taken equal to the Mean FL . Because therefore by the Hypothesis AB is to FL , as FL to BC , and O is equal to FL ; AB will also be to FL as FL to BC . Therefore (a) the Rectangle under the Extremes AB, BC , is equal to the Rectangle under the Mean FL and O , that is, is equal to the Square of FL .

(a) By the foregoing.

Part II. This is demonstrated in like manner from the second Part of the foregoing.

Corollary.

Fig. 24. **F**ROM this, taken together with the 13th, it is manifest that if in a Circle, FC be perpendicular to the Diameter, the Rectangle ACB is equal to the Square of FC .

(2.) If $A \times B$ be equal to the Square of C ; then $A :: C : B$.

(3.) If $A : C :: C : B$; and Cq be divided by A , the Quotient (b) will be B .

(b) Per

Corol. 2.

p. 16. l. 6.

P R O

PROP. XVIII. Problem.

UPON a given right Line (RS) to describe a Polygon like, and in like manner posited to given one (BQ.) Fig. 35.

Resolve the given Polygon BQ into Triangles. Upon the given right Line RS, make the Angles (a) R, O, equal to the Angles B A. The Sides then will meet together in (a) Per 23. l. 1.

Upon XS make the Angles V, I, equal to the Angles C. The Sides then will meet together in Z. I say, the Thing is done.

For because the Angles RO, are equal to the Angles A, the Angles E, K, must also be equal (*per* Corol. 9. 32. l. 1.) and because also by the Construction, V is equal to I, the whole EV must be equal to the whole KT. In the same manner because O, I, are equal to A, C, respectively, the whole Angles OI, AC must be equal. And because V and I also are equal to T and C by the Construction, ZQ likewise must be equal (*per* Cor. 9. p. 32. l. 1.) to T and C. Therefore the Polygons RZ, BQ, are mutually Equiangular. It remains, that we shew that their Sides also are proportional. RS is to BF, * as SX to * Per 4. l. 6.
 SX; and again, SX is to FL (b), as SZ to FQ. Therefore *ex æquo* RS is to SZ, as BF to FQ, &c. (b) By the same.

Corollary. "Hence is derived the Method of making Maps or Charts, whether Geographical, or Chorographical, or those which Surveyors of Land make; and of framing Ichnographical Delineations of Fields, Buildings, Countries: for they are nothing else but the Reduction of great Figures unto like Figures which are of a small Compass, which is performed by the means of this Proposition.

PROP. XIX. Theorem.

THE Proportion of like Triangles (X, Z,) is duplicate of the Proportion of their Sides (C, FI) which are subtended to the equal Angles. Fig. 36, 37

* *Per* 11. That is, if it be made * as AC is to FI , so is FI to third, AQ ; the Triangle X is to the Triangle Z , as AC to the first, to the third Proportional, AQ . See *Definitio* 10. 5.

Because the Triangles XZ , are like, BA will be to D (c) *Per* 4. (c) as AC is to IF . But by the Construction, as AC is to IF , so is IF to AQ . Therefore also BA is to D l. 6.

(d) *Per* 15. (d) as IF to AQ . Therefore in the Triangles QBA and Z , the Sides about the Angles A, I , (which, by the Definition of like Triangles, are equal) are reciprocal. Therefore

(e) *Per* 1. QBA and Z are equal (e). *But the Triangle X is to Z , as the Base AC to the Base AQ (f). Therefore

(f) *Per* 1. X is to Z , as AC to AQ . *Q. E. D.*

l. 6.

Corollary. "Hence is their Error to be corrected, who think that like Figures are in the same Proportion to one another, that their Sides are. For if of two, not only like Triangles, but also Squares, Pentagons, Hexagons &c. yea, and Circles also, the Sides or Diameters betwixt themselves, as 2 to 1, the Figures or Areas themselves are as 4 to 1. If the Sides be betwixt themselves, as 3 to 1, the Figures themselves or Areas, as 9 to 1; to wit, in a duplicate Proportion of the Sides.

PROP. XX. Theorem.

Fig. 38. LIKE Polygons ($ABCDE, FGHIK$) divided, (1.) Into like Triangles (P, S, Q, T , and R, V) in Number equal. (2.) *Proportional to the Wholes.* And (3.) The *Portion of the Polygons is duplicate to that of Sides, (AB, FG) which are betwixt the equal Angles (B, G , and BAE, GFK .)*

Part I. Because the Polygons are alike, they are mutually equiangular, *per Def. 1. l. 6.* Equiangular, and their Angles equal. BAE to GFK , and B to G , and BCD to GHI , and CDE to HIK , and E to K . Because therefore AB is to FG , as BC to GH , and the Angles B and G are equal, the Triangles P, S , (b) are like. In like manner it will be demonstrated, that R and V are like. Then, because the Wholes, BCD, GHI , and the subducted ones, BAE, GFK , are equal, the remaining

(a) *By the* BC (a) as FG to GH , and the Angles B and G are equal, the Triangles P, S , (b) are like. In like manner it will be demonstrated, that R and V are like. Then, because the Wholes, BCD, GHI , and the subducted ones, BAE, GFK , are equal, the remaining

(b) *Per* 6. demonstrated, that R and V are like. Then, because the Wholes, BCD, GHI , and the subducted ones, BAE, GFK , are equal, the remaining

l. 6. Wholes, BCD, GHI , and the subducted ones, BAE, GFK , are equal, the remaining

H F, are equal, the remaining ones also, A C D, F H I, are equal. In the same manner I might shew that A D C, I H are equal. Therefore (*per* Corol. 9. p. 32. l. 1.) the third C A D is equal to the third H F I. Where also (*e*) (*e*) *Per* 4. The Triangles Q and T are alike. The first Part thereof is manifest.

Part II. Because P and S are alike, the Proportion of P to S is duplicate to that of (*f*) C A to H F. But for the (*f*) *By the same Cause* also the Proportion of Q to T is duplicate to the Proportion of C A to H F. Therefore P is to S as Q is to T. In the same manner, I will shew that as Q is to R, so R is to V. Therefore, as one Antecedent, P, is to the Consequent, S, so all the Antecedents, P, Q, R, taken together, are to all the Consequents, S, T, V, taken together; that is, so is Polygon to Polygon. Which was the other Part.

Part III. The Proportion of P to S is duplicate (*h*) to (*h*) *By the* that of A B to F G. But the Proportion of Polygon to Polygon is the same with the Proportion of P to S, as I have already shewed. Therefore also the Proportion of Polygon to Polygon, is duplicate to the Proportion of A B to F G. Which was the third Part.

Corollaries.

ALL ordinate or regular Figures, as Squares, Equilateral Triangles, Pentagons, &c. are betwixt themselves in the duplicate Proportion of the Sides. For all regular Figures are like, as is manifest from *Definition* 1. 6.

2. If any like Figures whatsoever, the Sides A B, F G, which are placed betwixt equal Angles, be known, the Proportion of the Figures is also known. As for Example; let A B be of two Feet, and F G of six Feet; and as 2 is to 6, so let 6 be to some other Number; to wit 18. The lesser Figure is to the greater, as 2 is to 18, or as 1 is to 9. Now a third proportional Number is found, if (*per* Corol. 3. l. 6.) the second of the given ones be multiplied by itself, and the Product divided by the first.

3. From the same Proposition is drawn the excellent Method of encreasing or diminishing any Rectilineal Figure in given Proportion. As if I would make a Pentagon, whose Side is A B, five fold of another. Find a mean Proportional, B X, (*i*) betwixt the Terms of the Proportion.

K 3

given

(a) *Per* 18. given, AB, BC ; upon this Frame, (a) a Pentagon like
l. 6. to the given one. This shall be quintuple of the given
one.

For by the 20th, the Pentagon AB is to BX , which
like to it, as AB , the first, is to BC , the third Proportional.

Moreover, seeing the Proportion of Circles also is duplicate to the Proportion of their Diameters, as will be shewed, *p.* 2. l. 12. This Practice belongs likewise to Circles.

Fig. 41.

[Scholium. "Seeing the Proportion of the Squares E, K , is duplicate of the Proportion of their Sides OR, SV ; from thence the duplicate Proportion of the Sides OR, SV , is wont commonly to be expressed by the Proportion of OR q to SV q .]

PROP. XXI. Theorem.

Fig. 40.

Figures $A, (B)$ which are like to the same (C) are also like betwixt themselves.

This is manifest from *Definition 1. Lib. VI.* and from *Axiom 1. Lib. I.*

PROP. XXII. Theorem.

Fig. 40, 41.

If four or more right Lines (PI, LQ, OR, SV) be proportional; like Figures, and in like Sort described by them (AB and E) must also be proportional.

And Conversely.

The Demonstration of the first Part is manifest. For because the Proportions of A to B and E to K , are duplicate to the Proportions of PI to LQ , and OR to SV , which are, by the Hypothesis, equal, themselves also must be equal.

The second Part is manifest also.

Fig. 24.

[Corollary. "If the right Line AB be cut in any manner in C ; the Rectangle contained under the Parts AC

CB, is a mean Proportional betwixt their Squares. Likewise the Rectangle contained under the Whole AB, and one Part, AC or CB, is a mean Proportional betwixt the Square of the Whole, AB, and the Square of the said Part, AC or CB. For (*per Cor. 1. p. 8. l. 6.*) it is manifest, that $* AC : FC :: CF : CB$. There-
** Corol. 1.* fore AC Square : CF Square :: CF Square : CB Square. *p. 8. l. 6.*
 That is, $* AC \text{ Square} : \text{Rectangle } ACB :: \text{Rectangle } ACB : CB \text{ Square}$. *Q. E. D.*
** Per 17. l. 6.* Moreover, (*per Cor. 2. p. 8. l. 6.*) $BA : AF :: AF : AC$. Therefore $BA q : AF q :: AF q : AC q$. That is, $\dagger BA q : BAC \text{ Rectangle} :: BAC \text{ Rectangle} : AC q$. In the same manner $AB q : ABC :: ABC : BC q$. *Q. E. D.*

PROP. XXIII. Theorem.

Quiangled Parallelograms (*X, Z*) have betwixt themselves a Proportion that is compounded of the Proportions of their Sides (*AC, CB, and LC to CF.*) *Fig. 42.*

That is, if you make CB to be to O, as LC to CF, is to Z, as AC is to O.
 Let IL, SB. meet together in Q. The Parallelogram (*a*) is to the Parallelogram R, as AC is to CB; and R (*a*) *Per 1.* (*b*) to Z, as LC is to CF; that is, as CB is to O. *l. 6.*
 Therefore *ex aequo* X is to Z, as AC is to O. *Q. E. D.* (*b*) By the same.

Corollaries.

FROM hence, and from 34. l. 1. it is manifest.

1. That Triangles which have one Angle (at C) equal, have that Proportion betwixt themselves, which is compounded of the Proportions of the right Lines AC to CB, and LC to CF. Which Lines contain the equal Angle. *Fig. 42.*
2. That Rectangles, and consequently all Parallelograms whatsoever, have betwixt themselves the Proportion which is compounded of the Proportions of the Base to the Base, and the Height to the Height. And in the same manner the reason about Triangles.

3. Hence

- Fig. 42.* 3. Hence the Proportion of Triangles and Parallelograms may be readily learned. Let X and Z be the Parallelograms and their Bases AC , CB , and CL , CF , be their Heights.
- (c) *Per 12.* Let it be made (c) as the Altitude CL , is to the Altitude CF , so is one of the Bases CB , to O . The Parallelogram X is to the Parallelogram Z , as AC to O .

PROP. XXIV. Theorem.

- Fig. 43.* **I**N every Parallelogram (as S , F) the Parallelograms which are about the Diameter (AB) wit, (CL , OI) are both like to the whole Parallelogram, and to each other.

By 27. 1. the Angles, C , S , and L , F , are equal. E is equal to I , that is, by the same, equal to A it self; but B is common both to the Whole, S , F , and the Part, CL . Therefore the whole, S , F , and the Part CL , are Equiangular. It remains to be shew'd, that they have the Sides opposite to the equal Angles proportional.

Because in the Triangle BCE , BSA , CE is parallel to SA , BC (by *Corol. 1. p. 4. l. 6.*) will be to CE , as BS to SA : And CE will be to EB (by the same *Corollary*) as SA to AB . But because in the Triangles ELB , AFL , also, EL is parallel to AF , EB (by the same *Corollary*) will be to EL , as AB to AF . Therefore *ex æquo* CE to EL , as SA to AF . Therefore (by *Definition 1. l. VII*) CL , and the Whole, CF , are like. In the same manner I might shew OI to be like to the Whole, S , F . Therefore (per 21. l. 6.) CL and OI are also like betwixt themselves. *Q. E. D.*

PROP. XXV. Problem.

- Fig. 46.* **T**O change a given Polygon (A) into another like to a given one (B).
Or to make a Polygon equal to a given one (A) and like to another given one (B).

Upon CF , the Side of the Polygon B , a like one which is required, (by 45. l. 1.) make a Rectangle Q , equal to B . Then upon FI (by the same *Proposition*) make

Rectangle R equal to A. It is manifest, that C F and F I do make one right Line. Betwixt C F and F I find a mean Proportional F L (*a*). Upon this, (*b. 18. l. 6.*) make a (*a*) *Per 13.* Polygon like to the given one B, this must also be equal to *l. 6.* the given one A.

For seeing by the Construction, C F, F L, F I, are three Proportionals, the Polygon B is to the Polygon like to it, which is made upon F L, as C F is to F I (*per 20. l. 6.* and *Definition 10. l. 5.* that is, (*per 1. l. 6.*) as Q is to R. Therefore also by changing, as the Polygon B is to Q, so is the Polygon F L to R. But by the Construction, the Polygon B is equal to Q. Therefore also the Polygon upon F L, which is like to B, is equal to R; that is by the Construction, to the given A. That therefore is done which was required.

P R O P. XXVI. Theorem.

LIKE Parallelograms (B D, F N,) having a Fig. 44. common Angle (A) are about the same Diameter.

Draw the right Lines A E, C E. If you deny that A E C is a common Diameter to the Parallelograms B D and F N; let another right line A G C, which cuts F E in G, be the Diameter of B D, and draw the Parallel G H. The Parallelograms F H, B D, will be therefore about the common Diameter A G C, and consequently (*by 24. l. 6.*) will be like. Therefore, (*per Definition 1. l. 6.*) will be like. Therefore, as B A to A D, so is F A to A H. But also, as B A to A D, so is F A to A N, seeing B D, F N, are like by the Hypothesis. Therefore F A is to A H, as the same F A is to A N. Which is absurd.

P R O P. XXVII, XXVIII, XXIX.

THESE cause Trouble to, and perplex Beginners, and are scarce of any Use.

P R O P.

PROP. XXX. Problem.

Fig. 45. **TO** cut a given right Line (AB) so that the whole (AB) shall be to one Segment (AC) as the same Segment is to the Remainder (CB).

That is, as Geometricians speak, to cut a Line in extreme and mean Proportion.

By 11. L. 2. so cut AB in C , that the Rectangle under AB , CB , may be equal to the Square of AC . I say the Thing is done.

For by the 17th of this Book, as AB is to AC so is AC to CB .

The Force of this Section of a Line is admirable the inscribing and comparing regular Bodies.

PROP. XXXI. Theorem.

Fig. 47. **IF** from the Sides of a Rectangular Triangle (ACB) like Figures whatsoever be described that which is opposed to the right Angle, will be equal to the two others (L , R) taken together.

Here *Proposition 47. 1. 1.* is made universal.

From the right Angle C , let the Perpendicular CO be let down. Because (*per Corollary 2. P. 8. L. 6.*) AB , BO , are three Proportionals, F shall be to the Figure which is like to it, as AB the first, to BO the third Proportional, (to wit, by 20. L. 6. and *Definition 10. L.* Again, because (by the aforesaid Corollary) BA , AO , are three Proportionals, the Figure F shall (by the aforesaid *Proposition* and *Definition*) be to L , which is like to as BA the first, to AO the third Proportional. Because therefore F is to R , as AB is to BO ; and the same F is to L , as AB is to AO ; F shall also be to R and L taken together, as AB is to BO , AO , taken together. But A is equal to the two, BO , AO . Therefore also F shall be equal to the two, R and L . $Q. E. D.$

Corollary

Corollary:

FROM this Proposition we can easily find one Rectilinear Figure, equal and like to any Number of Rectilinear Figures whatsoever, by the same Method, whereby, *Prop. Schol. p. 47. l. 1.* one Square is found equal to any number of given Squares whatsoever. Only in the Demonstration, let 31. l. 6. be cited instead of 47. l. 1.

Corollary (2.) "A Circle upon the Hypothenufe of a Rectangle Triangle, is equal to two Circles described upon the Sides, for all Circles are like amongst themselves; and are to one another as the Squares of their Diameters, by the Second of the Twelfth Book.

Corollary (3.) "From hence we may derive that Quadrature of Lunets (or little Moons) which *Hippocrates of Chios* first taught. Qua-Fig. 54.

"For let ABC be a Rectangled Triangle; and BAC a Semi-circle to the Diameter BC: BNA a Semi-circle described on the Diameter AB; AMC a Semi-circle described upon the Diameter AC. Thus therefore the Semi-circle BAC is equal to the Semi-circles BNA and AMC together. If therefore you take away the two Spaces BA, AC, common on both Sides, there will be left the two Lunets BNA, AMC, bounded on both Sides with circular Lines equal to the Rectilineal Triangle BAC. And if the Line BA be equal to the Line AC, and you let fall a Perpendicular unto the Hypothenufe BC, The Triangle BAO will be equal to the Lunet BNA, and the Triangle COA equal to the Lunet CMA. Q. E. I.

PROP. XXXII.

THIS is hardly of any Use, and hath nothing remarkable in it.

PROP.

PROP. XXXIII. Theorem.

Fig. 48. **I**N the same or equal Circles, the Angles, whether at the Centers (as ABC , FOD) or at the Circumference (as ARC , FSD) have that Proportion betwixt themselves, which the Arches (AKC , FGD) on which they stand, have. Understand the same Thing of Sectors.

As for the Angles at the Center, and the Sectors, it will be demonstrated altogether in the same manner, in which *Prop. 1.* of this Book, it was demonstrated, that Triangles of the same Height are as their Bases: Only where *Prop. 38. l. 1.* is cited there, let *Prop. 29. l. 3.* be cited here.

And because the Angles R and S , at the Circumference are Halves of the Angles ABC , FOD , at the Center, that which hath been demonstrated of these will be manifest also of those.

Corollary.

Fig. 49. **T**HE Angle (BAC) at the Center, is to four right Angles, as the Arch BC on which it stands, is to the whole Circumference.

For as BAC is to the right Angle BAF , so by this, 3 the Arch BC is to the Quadrant BF . Therefore the Angle BAC is to four right Angles, as the Arch BC is to four Quadrants, that is, the whole Circumference.

The Arches IL , BC , of unequal Circles, which do subtend equal Angles, whether at the Center, as IAL and BAC , or at the Circumference, are like Arches.

For the Arch IL is (by *Corollary 1.*) to its Circumference, as the Angle IAL , that is, BAC is to four right Angles; and the Arch BC is to its Circumference (by the same *Corollary*) as the same Angle BAC is to four right Angles. Therefore IL is to its Circumference, as BC is to its. Therefore (by *Defn. 4. l. 6.*) the Arches IL and BC are like.

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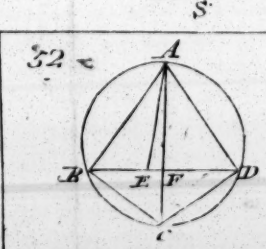
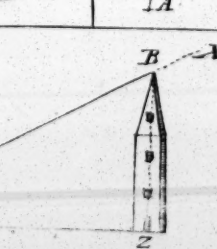
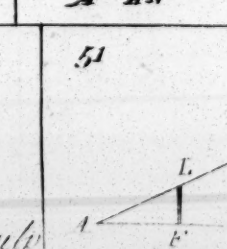
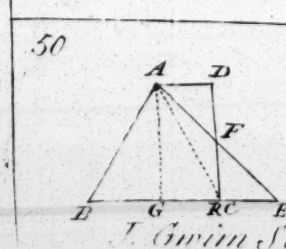
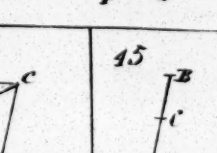
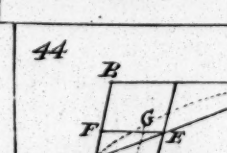
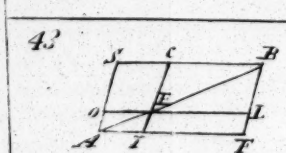
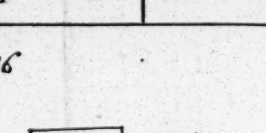
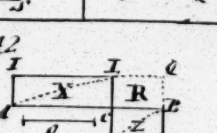
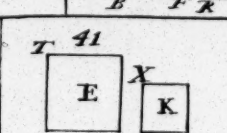
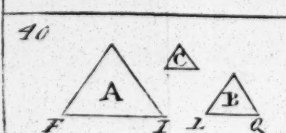
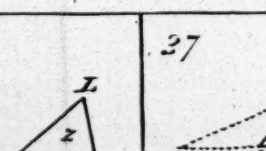
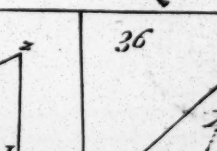
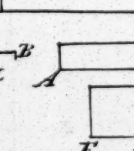
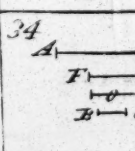
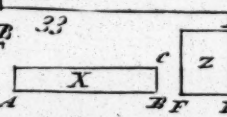
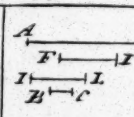
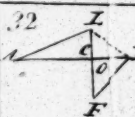
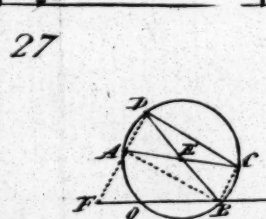
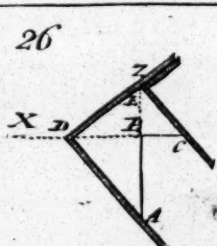
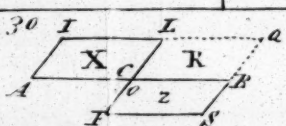
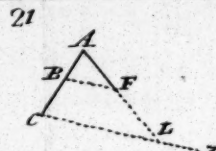
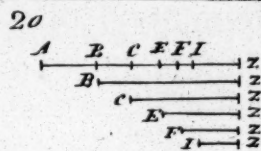
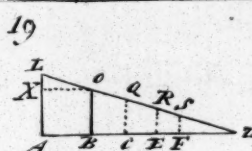
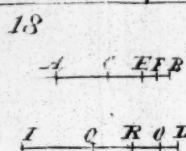
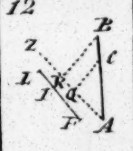
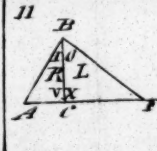
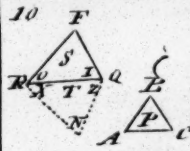
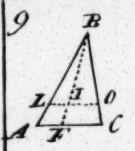
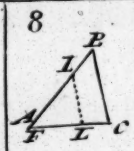
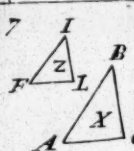
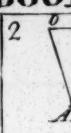
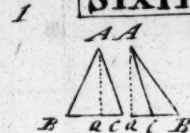
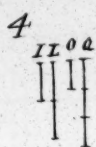
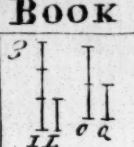
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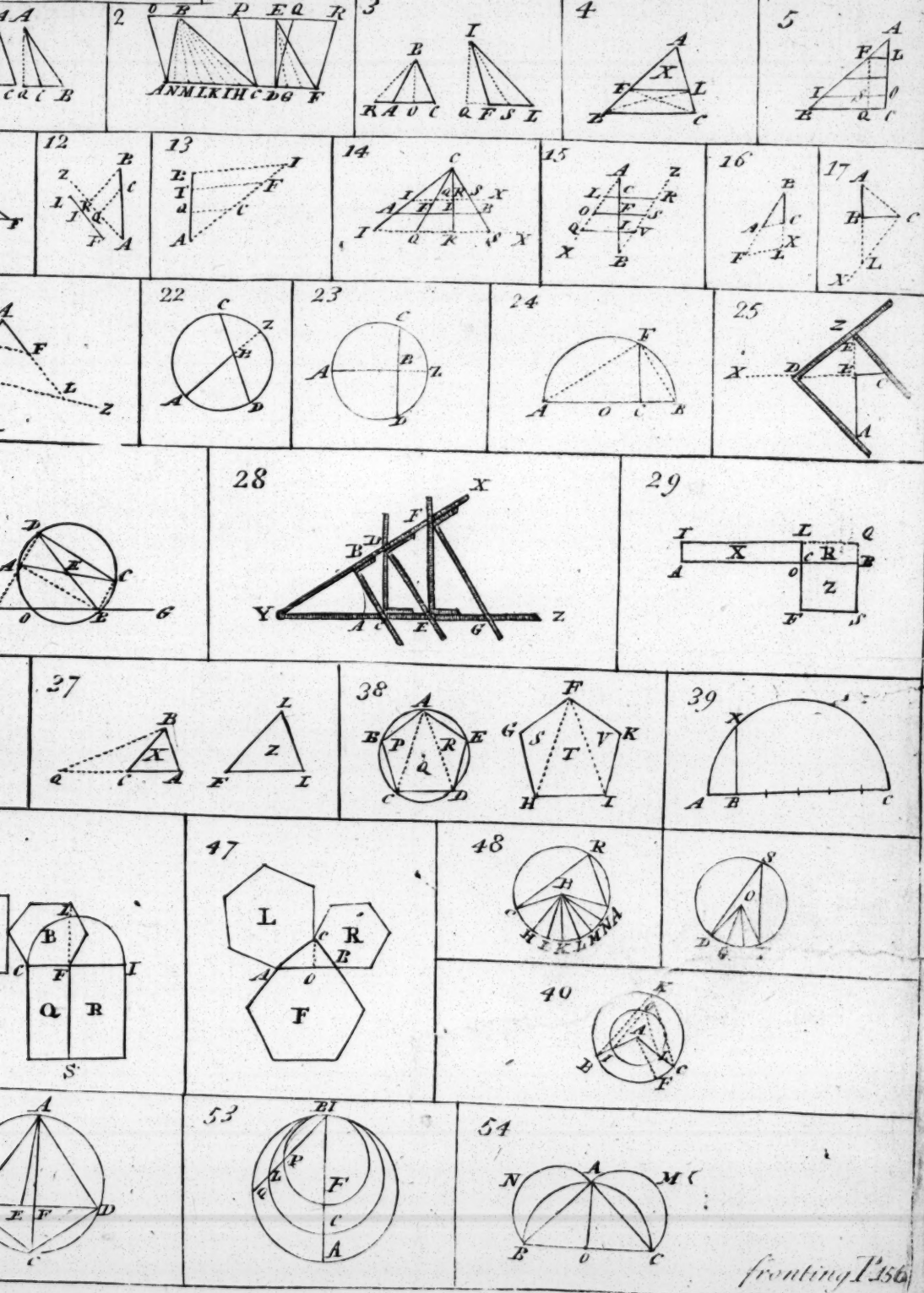
FIFTH BOOK

SIXTH BOOK



J. Gwyn Sculp

SIXTH BOOK



lib. VI.

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3. The Semi-diameters (A B, A C) do take away from concentrical Circumferences like Arches, I L, B C. This manifest from *Corollary 2*.

4. The Segments (B K C, I O L) which contain equal angles (K, O) are like.

For by *Corollary 2*. the Arches B C, I L, and consequent the Angles B K C, I O L, are like.





T H E

Elements of EUCLID

B O O K X I.

With Us the Seventh.

TO the six first Books *Euclid* subjoins the Elements of Numbers, comprehended in the three following the Seventh, Eighth and Ninth, to which he also adjoins a Tenth, concerning incommensurable Quantities. We pass immediately from Planes to Solids; purposing to treat of Numbers separately: Seeing it will, I suppose, be more commodious for Learners, if the Elements of Geometry be not interrupted, by treating of any other Matter, but be had altogether. Nevertheless, when we shall cite the Propositions of this and the following Book, we shall not call these Books the Seventh, and the Eighth, but the Eleventh and the Twelfth, lest if we should depart from the every where received Order of *Euclid*, the Citation of Propositions should thereby be rendered more intricate.

This Book in a sort contains two Parts: In the first, are laid the Foundation on which the whole Doctrine of the Solids, that is, of Bodies, depends. In the other, the Affections of Parallelepipeds are propounded.

“ This Eleventh Book of Elements sets forth the first Principles of Solids. Nor can indeed the Properties of Bodies be known without it; and if we set upon almost any Part of the Mathematicks, without the Knowledge of Solids, we shall labour in vain, or be at least at a great

“ Loss.

Lib. XI.

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Loſs. For the Spherical Doctrine of *Theodoſius*, Spherical Trigonometry alſo, a great Part of Practical Geometry, Statics and Geography, depends upon it; and what Things occur of any great Difficulty in the Art of Dialling, in the Conic Sections, Astronomy, Dioptrics or Optics, do all become more eaſy, the Principles of Solids being once underſtood. So that thoſe who have delivered the Elements of Geometry, leaving out and ſetting aſide this and the following Book, are to be reckoned to have delivered the ſame very imperfectly.

DEFINITIONS.

A Solid, or Body, is that which hath Length, Breadth and Thickneſs.

2. The Extreme of a Solid is a Surface.

Fig. 1. l. ii.

3. The right Line $[AB]$ is to the Plane $[CC]$ right perpendicular, when it makes right Angles $[BAC, BAC]$ with all the right Lines $[CA]$ in the Plane $[CC]$ by which it is touched.

4. A Plane is right or perpendicular to a Plane, when all the right Lines $[LQ]$ which are drawn in one of the Planes perpendicular to the common Section $[XR]$ are right or perpendicular to the other Plane $[ABCO]$.

Fig. 2.

5. If the right Line $[OL]$ ſtands upon a Plane not at right Angles, and from its higheſt Point $[L]$ there be drawn to the Plane the Perpendicular $[LP]$ and $[OP]$ be joined; the Angle $[LOP]$ is ſaid to be the Inclination of the Line $[OL]$ to the Plane.

Fig. 3.

6. If the Plane $[RE]$ doth not ſtand perpendicularly upon the Plane $[LQ]$ the Inclination of one to the other is the acute Angle $[ABC]$ which is contained by the right Lines $[AB]$ and $[BC]$ which are drawn in both Planes perpendicular to the common Section $[OE]$.

Fig. 4.

7. A Plane is ſaid to be alike inclined to a Plane, as is ſome other Plane to another, when the ſaid Angles of their inclinations are equal.

8. Parallel Planes, are thoſe which being continued every way, are always diſtant from each other by equal intervals.

9. Like ſolid Rectilinear Figures are thoſe which are contained under like Planes, in Number equal.

10. A ſolid right-lined Angle, is that which is contained under plain Angles more than two $[BAC, CAO, OAB]$ which

Fig. 5.

which are not in the same Plane, meeting together in one Point.

11. Equal solid Angles are those, which being conceived to be put each within the other, do agree or perfectly coincide.

Like as a plain Angle is a mutual Inclination of Lines, so a solid Angle is an Inclination of Surfaces. Concerning both therefore we must reason in the same manner.

Fig. 6, 7, 8. 12. A Prism is a solid Figure, comprehended by Planes amongst which two opposite ones [O F E, A C B] are parallel, equal and like.

Fig. 8. 13. A Parallelepiped is a Solid, contained under six Quadrilateral Planes, of which the Opposites are parallel.

14. If six Planes, in which the Opposites are parallel be Squares, the Solid contained by them will be a Cube.

PROPOSITION I. Theorem.

Fig. 9. **O**NE Part (A C) of a right Line cannot be in a Plane (O E;) and another Part (C B) out of it.

It is clear of itself, from the Definition of a Plane and a right Line. See *Defn.* 4. and 7. *L.* 1.

PROP. II. Theorem.

Fig. 10. **E**VERY Triangle is in one Plane: And two right Lines cutting each other, are in the same Plane.

For if a Plane be applied to one of its Sides, and to the Point of meeting the other two, it will be evident that the whole Triangle is in that Plane.

PROP. III. Theorem.

Fig. 11. **I**F two Planes (A B, C D) cut each other (E F) their common Section is a right Line.

It is manifest from the Definition of a Plane.
But we may demonstrate it thus. If EF, the common
section, be not a right Line, let there be drawn in the
Plane CD the right Line EOF, and in the Plane AB, the
right Line EQF. The two right Lines therefore, EOF,
QF, will inclose a Space. Which is absurd.

PROP. IV. Theorem.

If a right Line (BA) be perpendicular to two right Lines (CAX, FAS) which cut each other, will also be perpendicular to the Plane which is
Fig. 12.
own through them.

If you deny it, let another right Line, BQ, be perpen-
dicular to the Plane of the right Lines AC, AF. Join
Q, and to this in the Plane FAC, draw the Perpendi-
cular QO. This being produced, will necessarily cut (as is
inferred from *Schol. Prop. 31. l. 1.*) one of the right Lines
CAX, FAS, or both, wheresoever the Point Q shall be.
Therefore let it cut CAX in O, and let BO be joined.
Because therefore the Angle BAO is, by the Hypothesis,
a right one;

The Square of BO shall

be equal to BA Squ.

+ AO Squ.

(b.)

(b) Per 47.
l. 1.

but because BQ is supposed perpendicular to the Plane
AC, and consequently (by *Definition 3. l. 11.*) makes a
right Angle BQA with AQ;

BA Squ. is equal to BQ Squ.

+ AQ Squ.

(d.)

(d) Per 47.
l. 1.

and because the Angle AQO is, by the Construction,
a right one;

AO Squ. is equal to OQ Squ.

+ AQ Squ.

(e.)

(e) By the
same.

Therefore BO Squ. is equal to

+ BQ Squ.

+ OQ Squ.

+ AQ Squ.

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Therefore Square BO is greater than the Squares of BA and OQ ; and (as is clear from *Prop. 47. l. 1.* consequently BQO is not a right Angle. Therefore BQ is not perpendicular to the Plane by *Definition 3. l. 11.*) CA is perpendicular to the Plane. Therefore the Proposition is manifest.

Scholium.

FROM its being supposed that BQ is perpendicular to the Plane FAC ; it is directly demonstrated that CA is not perpendicular to that Plane; and consequently from the denial of the Assertion of the Theorem, the same Assertion is directly proved. This Demonstration, as to Substance of it is *John Cierman's*.

PROP. V. Theorem.

Fig. 13. IF three right Lines (BA, CA, FA) be perpendicular to the same right Line (AR) at the same Point (A ;) those three will be in one Plane.

For, if it may be, let one of them BA be in another Plane (RO) which may cut LQ , the Plane of the other two, CA, FA , in the right Line AO . Because, by the Hypothesis, RA stands perpendicularly upon the two, CA, FA , it will be perpendicular to the Plane LQ by the same Reasoning. Therefore RA makes a right Angle with AO (by *Definition 3. l. 11.*) But also, by the Hypothesis, RAB is a right Angle. Therefore the Angles RAO and RAB are equal. Which is absurd.

PROP. VI. Theorem.

Fig. 14. RIGHT Lines (AB, CD) which are perpendicular to the same Plane (CF) are parallel.

It might be taken for granted, as a Thing of course known; but we may demonstrate it thus:

BD being joined, make in the Plane FE the Line DG perpendicular to BD, and equal to BA; and let DA, A, GB, be joined. The right Lines BD, DG, are equal to BD (a) and BA; and the Angles BDG, (b) (a) By the BA are right ones. Therefore (per 4. l. 1.) AD, BG, Construction. equal. Therefore the Triangles ABG, GDA, are equal. Therefore the Angles BGD, GDA, are equal. But ABG (by Defn. 3. l. 11.) Def. 3. l. is a right Angle. Wherefore ADG is also a right one. 11. BDG also, by the Construction, and CDG, by Defn. 3. are right Angles. Therefore GD is perpendicular to the three Lines CD, AD, BD. Therefore CD is in one Plane with AD and BD. But AB also is in (c) By the Plane (per 2. l. 11.) with AD and BD. Therefore foregoing. AB, CD are in one Plane. Therefore seeing the Angles BD, CDB (by Defn. 3. l. 11) are right ones, AB, CD, will (per 29. l. 1. and Defn. 36. l. 1.) be parallel. Q. E. D.

PROP. VII. Theorem.

Right Line (EF) cutting the right Lines Fig. 15. (AB, CD) placed in the same Plane, is in one the same Plane with them.

It might be taken for granted. But he that will may demonstrate it. Let another Plane cut the Plane of the right Lines AB, CD, in the Points EF. If now EF is not in the Plane of AB, CD, EF will not be the common Section. Let EGF therefore be so. Therefore (per 3. l. 11.) EGF is a right Angle; the two right Lines therefore EF, EGF, inclose a Space. Which is absurd.

Corollary.

THENCE it follows, that if EF cut the Parallels AB, CD, it is in the same Plane with them. For (by Definition 36. l. 1.) any two Parallels are in the same Plane.

PROP. VIII. Theorem.

Fig. 14. **I**F two Parallels (AB, CD) one (AB) be perpendicular to a Plane EF ; the other also (CD) will be perpendicular to the same Plane.

It might be taken for granted. If the Demonstration be required, it is as follows:

“ BD, AD , being drawn; in the Plane EF , make
 “ GD perpendicular to BD . It will also (see the Demonstration of *Prop. 6. l. 11.*) be perpendicular to AD .
 “ Therefore (*per 4. l. 11.*) GD will be perpendicular to the Plane ABD , that is, (by the foregoing Corollary) to the Plane $CBDA$. Wherefore (*per Defn. 3. l. 11.*) CDG is a right Angle. But the Angle CDB is also a right one; forasmuch as with ABD , which (*per Defn. 3. l. 11.*) is a right Angle, it maketh two right Angles. Therefore (*per 4. l. 11.*) CD is perpendicular to the Plane GDB or EF . *Q. E. D.*

PROP. IX. Theorem.

Fig. 16. **R**IGHT Lines (AB, EF) which are parallel to the same right Line (CD) although they be not in the same Plane with it, are also parallel to each other.

Although it might be taken for granted, yet we will demonstrate it thus:

In the Plane of the Parallels AB, CD , draw GK perpendicular to CD . Likewise in the Plane of the Parallels EF, CD , draw HK perpendicular to CD . There

- (a) *Per 4. l. 11.* CK is perpendicular to the Plane GKH . Therefore seeing AG, EH , be parallel to CK , the same AG, EH will be perpendicular to the Plane GKH . Therefore (*per 8. l. 11.*) AG, EH are parallel. *Q. E. D.*
- (c) *Per 6. l. 11.*

PROP. X. Theorem.

If two right Lines (AC , BC) be parallel to two right ones (DE , EF ;) albeit they be not in the same Plane, they comprehend equal Angles (C and

Fig. 17.

Let CA , CB , be made equal to FD , EF , and let DE , DA , FC , EB , be drawn. Seeing AC , FD , are parallel and equal, AD also and CF will (a) be parallel and equal. In like manner I might shew BE , CF , to be parallel and equal. Therefore AD , BE , are also parallel and equal, per Axiom 1. Therefore, per 33. l. 1. AB , DE , are equal. Seeing therefore the Triangles BAC , EDF , are Equilateral to each other, the Angles C and F are equal. Q. E. D.

(a) Per 33.

l. 1.

(b) By the foregoing.

(c) Per 8.

l. 1.

PROP. XI. Problem.

To draw a Perpendicular to a given Plane (AB) from a Point given without it (C .)

Fig. 18.

The Construction. In the Plane AB , draw any right line, as DF , unto which, from C , erect the Perpendicular CE . Then in the Plane AB , through E , draw AE perpendicular to the same DF . Then to AM , from C , draw the Perpendicular CG . I say, that CG is perpendicular to the Plane AB .

Through G let HG be drawn parallel to DF . By the construction, DE is perpendicular to CE and EM . Therefore DE is perpendicular to the Plane CEM (d), as (d) Per 4. l. 11. is HG (e). Therefore, by Defin. 3. l. 11. CG is perpendicular to HG . But CG , by the Construction, is perpendicular to EM . Therefore (f) CG is perpendicular to the Plane AB . Which was the thing proposed. (f) Per 4. l. 11.

Scholium. " In Practice thus. Let there be a Cord or Rule fastned to the given Point A : And from the same, let there be described by the end of it B in the Plane given, the Circle $BCFL$. The Line AK , which connects the given Point and the Center of the Circle, will be perpendicular to the given Plane.

Fig. 20.

l. 12.

PROP. XII. Problem.

Fig. 19.

FROM a given Point (*A*) in any Plane (*E F*) to erect a Line perpendicular to the same Plane.

From any Point *D*, without the Plane *E F*, make *D B* (by the foregoing) perpendicular to the Plane *E F*. And *B A* being joined, draw *A C* parallel to *D B*. I say the Thing is done. The Demonstration is manifest from *Prop. 8*.

Corollary.

IN Practice, from the given Point, a Perpendicular is erected to the given Plane, if a Square *OK N* be applied to the given Point [and be turned round.]

PROP. XIII. Theorem.

Fig. 20.

LINES drawn from the same Point cannot be both perpendicular to the same Plane (*A B*).

For if they were, they would, (by *Prop. 6*.) be parallel. Which cannot be.

PROP. XIV. Theorem.

Fig. 21.

IF the same right Line (*A B*) be perpendicular to two Planes (*F G*, *L Q*;) the Planes will be parallel.

Let there be taken in either of the Planes, as *F G*, Point *C*, from which let *C E* be drawn parallel to *A B* and meeting the Plane *L Q* in *E*. Then *C E* (per 8. l. 1) will be perpendicular to both Planes, *F G*, *L Q*. Wherefore if *A C*, *B E* be joined, the Angles *A B*, (by Def. 11) will be right ones. Therefore per 29. l. 1) *A B*, *B E*, are parallel. Therefore *A C E B* is a Parallelogram and consequently *C E*, which hath been already shewn

perpendicular to both Planes, is equal (*per* 34. *l.* 1.) to
 B. In the same manner I might shew that all the Per-
 pendiculars to both Planes are equal. Therefore (by *Defin.*
 1. 11.) the Planes are parallel. Q. E. D.

PROP. XV: Theorem.

If two right Lines (*B A, C A*) touching each other be parallel to two right Lines which
 do touch one another (*E D, F D*;) the Planes
 likewise which are drawn through them will be
 parallel.

From *A*, let there be drawn *A G*, perpendicular to the
 line *E F*, and let *G H, G I*, be parallel to *D E, D F*.
 These (*per* 9. *l.* 11.) will also be parallel to *A C, A B*.
 Being therefore the Angles *I G A, H G A*, be, by *Def.* 3.
 right; *C A G, B A G*, will also (a) be right Angles. (a) *Per* 2.7.
 Therefore, *G A*, which is perpendicular to the Plane *E F*, *l.* 1.
 will also be perpendicular to the Plane *B C* (b). Therefore (b) *Per* 4.
 the Planes *B C, E F*, are, by the foregoing, parallel. *l.* 11.
 Q. E. D.

PROP. XVI. Theorem.

Plane (*E H F G*;) cutting parallel Planes Fig. 23.
 (*A B, C D*;) makes the Sections in them,
 (*H, G F*) parallel.

If not, seeing they be in the same intersecting Plane,
 they will meet somewhere, by *Schol. Prop.* 21. *l.* 1. as in *I*.
 Therefore seeing the whole Lines *H E I, F G I*, be in the
 Planes * *A B, C D*, produced, these Planes also will meet in * *Per* 1.
 Which is absurd, and contrary to *Defin.* 8. *l.* 11. *l.* 11.

PROP.

PROP. XVII. Theorem.

Fig. 24. **P**arallel Planes cut right Lines ($B D$ and $G H$) proportionally.

Let the right Lines $B H$, $G D$, be drawn in the Plane $P V$, $T Q$; and likewise let $B G$ be drawn meeting the Plane $R S$ in F , and let $F C$, $F I$ be joined. The Plane $B G D$ cutting parallel Planes, makes the Sections $C F$, $D G$ parallel, by the foregoing. Therefore $B F$ is to $C D$, as $B F$ (e) to $F G$. Again, the Triangle $B H I$ cutting parallel Planes, makes the Sections, by the foregoing, $B H$, $F I$, parallel. Therefore $H I$ is to $I G$, as $B F$ to $F G$; that is, as I have already shewed, as $B C$ to $C D$. *Q. E. D.*

(e) *Per 2.*
l. 6.
* *Per 2.*
l. 6.

PROP. XVIII. Theorem.

Fig. 25. **I**f a right Line ($F E$) be perpendicular to a Plane ($A B$), all the Planes which are drawn through it are perpendicular to the same Plane ($A B$).

Let the Plane $G C$ be drawn through $F E$, making $C D$ the common Section with $A B$; and let the Lines $H K$ be drawn in the Plane $G C$, perpendicular to the common Section, $C D$. Now seeing, by the Construction, $H K$ perpendicular to the same common Section to which $F E$ is perpendicular, by the Hypothesis, $H K$ and $F E$ must be parallel, by 29. l. 1. Therefore $H K$ is also perpendicular to the Plane $A B$, per 8. l. 11. Therefore the Plane $G C$ is perpendicular to the Plane $A B$, per Definition 4. l. 1.

PROP. XIX. Theorem.

Fig. 26. **I**f two Planes ($M F$, $G D$) cutting each other be both perpendicular to the same Plane ($A B$), their common Section also will be perpendicular to that Plane ($A B$).

For seeing, by the Hypothesis, the Plane MF is perpendicular to the Plane AB ; it is manifest, by *Definition* 4, that there may be drawn in the Plane MF , from the Point L , a Perpendicular to the Plane AB ; namely, that which from L , in the Plane MF , is perpendicular to the common Section EF . Again, by the Hypothesis, GD is perpendicular to that Plane AB ; 'tis evident, in the Plane GD , may be drawn from the Point L , a Perpendicular to the Plane AB . But from the Point L (*a*) there can be (*a*) *Per* 13. erected only one Perpendicular to the same Plane AB . *l.* 11. Therefore the Perpendicular to the Plane AB , which is drawn from the Point L , must be found in both the Planes, MF and GD , and consequently LK , the common Section of those two Planes. MF and GD , is perpendicular to the Plane, AB . *Q. E. D.*

PROP. XX. Theorem.

If a solid Angle (A) is contained under three plain Angles (BAC , CAD , DAB ;) any two of these is greater than the third. *Fig. 27.*

If the three Angles be equal, the Assertion is manifest at first Sight; and it is as certain, if they be unequal. For let BAD be the greatest; and from BAD , cut off BAE , equal to BAC , and make the Line AC equal to AE . And through E , let there be drawn a right Line meeting AB and AD , in B and D , and let BC , DC , be joined. Because, by the Construction, the Angles BAE , BAC , are equal, as likewise the Sides BA , AE , equal to the Sides BA , AC , the Bases also BE , BC , will be equal (*b*). And because BC , CD , (*c*) are greater than BD , the Equal, BE , BC , being taken away, there remains CD greater than ED . But the Sides EA , AD , are, by the Construction, equal to the Sides, CA , AD . Therefore the Angle (*d*) CAD is greater than the Angle EAD . Seeing therefore the Angle BAC is equal, by the Construction, to the Angle BAE , those two Angles together, BAC , CAD , are greater than the Whole, BAD . *Q. E. D.*

PROP.

PROP. XXI. Theorem.

THE plain Angles constituting any solid Angle whatsoever, are less than four right ones.

Fig. 28.

Let A be a solid Angle; let the right Lines, B C, C D, D E, E F, F B, be subtended to the plain Angles which make up the solid one, so that those right Lines be all in one Plane. Which being done, there is constituted a Pyramid, whose Base is the Polygon B C D E F; A is the Top, and it is contained under so many Triangles, G, H, I, K, L, as there are plain Angles which compose the solid Angle A. And now, because the two Angles A B F, A B C, are, by the foregoing, greater than the third, F B C; and the two, A C B, A C D, are greater than the third, B C D, and so on: All the Angles of the Triangles, G, H, I, K, L, about the Base, as taken together, are greater than all the Angles of the Base, B, C, D, E, F, taken together. But the Angles of the Base, together with four right ones, make twice so many right Angles, by *Theorem 2. Scholium* after 32. *L. 1.* as there are Sides, or, which is the same, as there are Triangles. Therefore all the Angles of the Triangles about the Base, together with four right ones, make more than twice so many right Angles as there are Triangles. But the same Angles about the Base, together with the Angles that compose the solid make up (a) twice so many right Angles as are the Triangles. It is manifest therefore, that the Angles which compose the solid Angle A, are less than four right ones. *Q. E. D.*

Corollary.

FROM this and the foregoing, it is obvious to collect that a solid Angle may be composed of any three plain Angles, which are less than four right ones, if so be that any two of them be greater than the other.

Scholium

Scholium.

FROM this Proposition is demonstrated that famous Theorem, That only three regular and equal plain Figures can contain a Body; to wit, Equilateral Triangles, either 4, or 8, or 20; 6 Squares, and twelve Pentagons. And consequently, that there are only five regular Bodies. A *Pyramid*, which is contain'd under 4; an *Octaedrum*, which is comprehended by 8; and an *Icosiedrum*, which is bounded by 20 Equilateral Triangles; a *Cube*, which is contain'd under 6 Squares; and the *Dodecaedrum*, under 12 regular and equal Pentagons. Now a Body is called Regular, which is comprehended under regular and equal Planes.

Demonst. A solid Angle cannot be compos'd of only two Equilateral Triangles; three, at least, are requir'd.

Of three Equilateral Triangles meeting in one Point, there may be constituted the solid Angle of a *Pyramid*; of four, the solid Angle of an *Octaedrum*; of five, the solid Angle of an *Icosiedrum*: Forasmuch, as both 3, 4, and 5 Angles of an Equilateral Triangle are less than 4 right ones, as is gathered from *Corollary 12. Proposition 32.*

1. And because three Angles of a regular Pentagon, as is gathered from *Corollary, Prop. 11. l. 4.* are less than four right ones, three Pentagons, meeting in one Point, will constitute a solid Angle, that of the *Dodecaedrum*.

That of the three Squares, meeting in one Point, may be compos'd the solid Angle of a *Cube*, is manifest of itself. And thus there arise five regular Bodies.

But that there are no more than these five, is thus proved. Six Angles of an Equilateral Triangle make just four right ones. For one is two Thirds of one right one; and therefore six such will make, by *Corol. 12. Prop. 32. l. 1.* twelve thirds of one right one, that is, four right ones. And therefore of six Equilateral Triangles a solid Angle cannot be compos'd, much less of more.

That of four Squares a solid Angle cannot be made, much less of more, is manifest in itself.

Four Angles of a regular Pentagon are greater than four right ones. For, by *Coroll. Prop. 11. l. 4.* each of them make

make six Fifths of one right one. Therefore a solid Angle cannot be made of four such Pentagons; much less of more.

Nor can a solid Angle be compos'd of any other regular Figures whatsoever. Three Angles of a regular Hexagon, by *Corollary 2. Prop. 15. l. 4.* are equal to four right ones. For one makes four Thirds of one right one; and therefore three make twelve Thirds of one, that is, four entire right ones. Therefore of three Hexagons a solid Angle cannot be made up; much less of more.

But seeing three Angles of a regular Hexagon are equal to four right ones, three Angles of any other Figure whatever greater than an Hexagon, as of an Heptagon, Octagon, &c. will be greater than four right ones. Wherefore it is manifest, that the rest of the regular Figures are all incapable of composing a solid Angle; and consequently, that there can be no regular Bodies besides the five foregoing.

PROP. XXII, XXIII.

ARE very prolix, and tedious to Beginners, and scarce at any Time come into Use.

PROP. XXIV. Theorem.

Fig. 29. **THE** Planes which contain a Paralleloepid are
(1.) Parallelograms. (2.) The opposite ones are like. (3.) The Planes are equal.

Part I. The Plane A F, cutting the Planes B D, F H, which by *Defn. 13.* are parallel, makes (a) the Section 16. l. 11. B A, F E, parallel. Again, the Plane A F, cutting the Planes A H, B G, which, by the same Definition, are parallel, (by the same) makes the Section A E, B F, parallel. Therefore B A E F is a Parallelogram. By the like Argument the rest of the Paralleloepid may be prov'd to be Parallelograms.

Part II. Because it is manifest from the first Part, that (b) *Per* A B, B C, are parallel to E F, F G; the Angles A B C 10. l. 11. E F G, must be (b) equal. Wherefore seeing the alternate Sides

Sides also are equal, the opposite Parallelograms $B D$, $F H$, are like or similar. And the same of the rest.

Part III. This is manifest from the first Part, and Fourth, or Eighth of the First Book.

PROP. XXV. Theorem.

IF a Paralleloepid ($G F D I$) or any Prism Fig. 30. whatever be cut by a Plane ($N P$) that is Parallel to the opposite Sides; there will be this Proportion, as the Base ($D C P O$) is to the Base ($O P F E$) so is the solid ($G P$) to the solid ($N F$.)

This is demonstrated in the same manner as *p. 1. l. 6*.

Corollary.

A Prism cut by a Plane parallel to the opposite Planes, hath a Section like and equal to the opposite Planes.

PROP. XXVI, XXVII.

ARE not necessary.

PROP. XXVIII. Theorem.

A Plane passing through the Diameters of opposite Planes ($A C$, $E G$) cuts the Paralleloepid into two equal Prisms. Fig. 29.

Because (*a*) $B G$, $B E$, are Parallelograms; $C G$, $A E$, (*a*) *Per* are equi-distant from the same $B F$. Therefore (*b*) they are 24. *l. 11*. also parallel betwixt themselves, and consequently are in (*b*) *Per* 9. one Plane. Therefore the right Lines $A C$, $E G$, are (*c*) *l. 11*. in one Plane. But now that a Plane drawn through them (*c*) *Per* 7. both cut the Paralleloepid into two equal Prisms, is thus *l. 11*. shew'd. Let the Prism $A E G C D H$ be understood to be constituted upon its Plane $A E C G$, that the Angles D , H , bend

bend towards the Angles B, F. It is manifest, that it will yet be betwixt the parallel Planes B A D C, F E H G. But then D must needs fall upon B, and H upon F. For let D fall without B, if it may be, and in N. The Angle (d) *Per* 27. B A C (*d*) is equal to the Angle D C A. But D C A is equal to N A C, for it is one and the same Angle. Therefore B A C and N A C are equal: Which is absurd. Therefore D falls upon B; and for the same cause, H upon F. Therefore the Prism A E G C D H coincides with the Prism A C G E F B, and consequently they are equal, by *Axiom* 7.

PROP. XXIX, XXX. Theorems.

Fig. 31. **THE** *Paralleloepids* (*FEAFKIMC*) and (*FEBHLOMI*) which have the same Base (*EFIM*) and the same Altitude, and consequently exist between parallel Planes (*EFIM*) and (*GAOL*) are equal.

For they either exist betwixt the lateral parallel Planes E A O M and F G L I, or not. Let the first be supposed from the 24th of this, and the 8th of the first Book, it is manifest, that the Triangles A E B, C M O; likewise G F H, K I L, are Equilateral and Equiangular to each other. Wherefore, as in the foregoing, I might shew that the Prisms C M O L I K, and A E B H F G, being laid upon each other will coincide, and consequently, by *Axiom* 7, are equal. Wherefore the common Solid F E B H K C M I being added, the whole *Paralleloepids* F E A G K I M C and F E B H L O M I are equal. *Q. E. D.*

Then let the *Paralleloepid* F X Q E M I P R not exist betwixt the same lateral parallel Planes with the *Paralleloepid* F E A G K C M I. Here, because, by the Hypothesis, G K, A C, R P, Q X, are in one Plane, which is parallel to the Base E F I M; let R P, Q X, cut G K in L and H, and A C in O and B; and let E B, M O, F H, I L, be join'd. It is easy now to shew, that the Planes containing the Solid F E B H L O M I, are *Parallelograms*, the opposite ones of which are equi-distant, and consequently that the Solid is, by *Defn.* 13. *L. 11.* a *Paralleloepid*. But to this, by the first Part, the *Paralleloepids* F X Q E M I P R, and

and $F E A G K C M I$, are each of them equal. Therefore they are also equal betwixt themselves. *Q. E. D.*

Corollary.

THIS Proposition is like to the 35th of the first Book; for it affirms concerning Solids, what that doth touching Planes. Wherefore the Demonstration of the rest of the Cases will be like also.

P R O P. XXXI. Theorem.

Paralleloepids upon equal Bases ($A O$ and $E G$) and in the same Altitude (S) are equal. Fig. 33.

First, let the *Paralleloepids* have their Sides perpendicular to the Bases. Unto the Side $F G$, produced, let there be made a Parallelogram $G M K H$, equal and like to the Parallelogram $A O$; and the Parallelogram $G M P R$ being perfected, let the right Lines $P M$, $R G$, meet $K H$ in Q and L . And now let *Paralleloepids* be understood to be constituted upon $G K$, $G Q$, $G P$, whose Sides are perpendicular to the Bases, and S is their common Altitude. The Solid $E G S$, is to the Solid $G P S$, as $E G$, *per 25. l. 11.* is to $G P$; that is, because $E G$, $A O$ are equal, by the Hypothesis, as $A O$ to $G P$; that is by the Construction, as $G K$ is to $G P$; that is, as $G Q$ is to $G P$, *per 5. l. 11.* that is, as the Solid $G Q S$ is to the same Solid $G P S$, *per 25. l. 11.* Because therefore the Solids $E G S$ and $G Q S$ have the same Proportion to the Solid $G P S$, the Solid $E G S$ will be equal to the Solid $G Q S$; that is, to the Solid $G K S$, *per 29. l. 11.* that is, because the Bases $G K$, $A O$, are equal and like, by the Construction, to the Solid $A O S$, as it appears from 29. *l. 11.* and even in itself. Which was the Thing propos'd. Note, That in this reasoning, the Solids are suppos'd to be right or perpendicular ones.

Then let the given *Paralleloepids* $E G S$, $A O S$, have their Sides at the Bases $E G$, $A O$, oblique. Let there now be made upon $E G$, $A O$, *Paralleloepids*, whose Sides are perpendicular to the Bases in the Height S ; these will be equal to the oblique ones by the 29th and 30th. Wherefore seeing, by the first Part, right *Paralleloepids* are

are equal betwixt themselves, the oblique ones will be equal betwixt themselves likewise. *Q. E. D.*

P R O P. XXXII. Theorem.

Fig. 34. **ALL** *Parallelopedids whatever of equal Height are betwixt themselves as their Bases.*

Let GO and A be the Bases. Upon CO make the Parallelogram OE equal to A .

Upon BC , OE , let Parallelopedids be understood to be erected in the Altitude K ; these therefore will be Parts of one Parallelopedid, BEK . Therefore the Parallelopedid OEK , is to the Parallelopedid, BCK , as the Base OE to the Base BC , *per 25. l. 11.* that is, by the Construction as the Base A is to the Base BC . But because the Bases OE and A , are equal, the Parallelopedids, OEK and AK , are equal, by the foregoing. Therefore also the Parallelopedid, AK , is to the Parallelopedid, BCK , as the Base A is to the Base BC . *Q. E. D.*

Scholium.

THAT which hath here been shew'd of Parallelopedids will be demonstrated in the Twelfth Book of Pyramids, *Prop. 6.* Of all Prisms whatever, in *Corollary* after *Proposition 9.* Of Cones and Cylinders, *Proposition 11.*

P R O P. XXXIII. Theorem.

Fig. 35. **LIKE** *Parallelopedids (HA and CM) are in a triplicate Proportion of their homologous Sides (AB , BC .)*

Let the Parallelopedids, AH , CM , be like. Therefore all their Planes, by *Defn. 9. l. 11.* are like; and consequently AE , by *Defn. 1. l. 6.* is to BC , as EB to BO and as FB is to BG , so is EB to BO . Moreover the Angles of the Planes are also equal, by the same. Therefore, let the Solids AH , CM , be so placed, that the equal Angles CBO , ABE , may be opposite, and the

des AB , C
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Lines A B, C B, may lie so as to make one strait Line ;
 and then E B, O B will also lie so as to make one strait Line.
 Now, let Solids be imagin'd to be constituted upon the
 Planes B Q and E C, in such sort that the Solids K B, H A,
 may be one Paralleloepid, and K B, P O, may make one
 Paralleloepid, and P O, C M, may make one Paralleloepid
 likewise. The Solid H A, is to the Solid K B *per* 25.
 11. as A E to B R ; that is, *per* 1. l. 6. as A B to B C ;
 that is, as I shew'd above, by the Hypothesis, as B E is
 to B O ; that is, by the same, as E C is to B Q ; that is,
per 25. l. 11. as the same Solid K B, is to the Solid P O.
 Therefore the three Solids, H A, K B, P O, continue the
 same Proportion. But now the Solid K B, is to the Solid
 P O, by the same, as the Base B R, is to the Base B Q ;
 that is, *per* 1. l. 6. as E B is to B O ; that is, as F B is to
 B G, as it was shew'd above, by the Hypothesis ; that is,
 the same, as the Plane F C is to the Plane B S ; that is,
per 25. l. 11. as the same Solid P O again is to the Solid
 C M. Therefore the four Solids, H A, K B, P O, C M,
 are continually proportional. Therefore, by *Defin.* 10. l.
 the Proportion of the first H A, to the fourth C M,
 is triplicate of the Proportion of the first H A, to the se-
 cond K B ; that is, triplicate to the Proportion, *per* 25.
 11. of A E to B R ; that is, triplicate, *per* 1. l. 6. to
 the Proportion of the homologous Sides, A B to B C.
 E. D.

(Corollary (1.) " Hence, if there be four right Lines
 continually proportional ; as is the first to the fourth,
 so is a Paralleloepid describ'd upon the first, to a Paral-
 leloepid like, and in like manner describ'd upon the
 second.

(2.) " Upon this also depends that most famous Pro-
 blem concerning doubling the Cube ; of which after-
 wards, *Scholium*, p. 18. l. 12.

(3.) " Hence also is to be corrected the Error of those,
 who suppose that the Proportion of like Solids is the
 same as is that of their Sides. For the Cube of a Line,
 which is double to another Line, is not only double to
 the other, but as eight to one. And the Cube of a
 Line, which is treble to another Line, is not only treble
 to the other Cube, but contains it 27 Times For 1 :
 2 : 4 : 8 ÷ and 1 : 3 : 9 : 27 ÷, and the same thing is

“ to be said of all like Bodies whatsoever ; as will app
“ afterwards.

(4.) “ Hence the triplicate Proportion of any Qua
“ ties whatsoever is the Proportion of the Cubes of
“ same Quantities. Let there be any two Quantities in
“ triplicate Proportion of the Quantities, AB, BC ;
“ shall also be as AB Cube, is to BC Cube.]

Scholium.

THAT which hath here been shew'd of Paralleloep
will be demonstrated, Book 12. Of Pyramids, P
8. Of all Prisms whatsoever, *Corollary 2. Prop. 9.*
Cones and Cylinders, *Prop. 12.* Of Spheres, *Prop. 1*

PROP. XXXIV. Theorem.

Fig. 36.

IF the Paralleloepids (BM, CK) be equal, th
Bases and Altitudes are reciprocally proportion
(that is, the Base AM is to the Base FK , as r
procally the Height FC is to the Height AB .)
And if their Bases and Altitudes be reciproc
proportiona^l, they are equal.

Part I. First, let the Sides be perpendicular to the B
If now the Altitudes of the Solids, BM, CK , be eq
the thing is manifest.

If the Altitudes be unequal, from the greater, FC ,
off FE , equal to BA ; and through E draw the Plane
parallel to FK . The Base AM , is to the Base FK ,
25. *l. 11.* as the Solid BM , is to the Solid EK ; tha
because, by the Hypothesis, the Solids BM, CK
equal, as the Solid CK , is to the Solid EK ; that is,
the same, as CG is to EG ; that is, *per 1. l. 6.* as C
to EF ; that is, by the Construction, as CF to B
Q. E. D.

Then let the Sides be oblique to the Bases. Let r
Paralleloepids be erected upon the same Bases in the s
Height. The oblique Paralleloepid will be equal to th
Wherefore seeing these, by the first Part, have their B
and Altitudes reciprocal, those also will be like
Q. E. D.

Then let the Sides be oblique to the Bases. Let right Paralleloepipeds be erected upon the same Bases in the same height. The oblique Paralleloepipeds will, *per* 29. and 30. be equal to these: Wherefore seeing these, by the first, have their Bases and Altitudes reciprocal, those also will be so likewise. *Q. E. D.*

Part II. Let the Altitudes be unequal, and the Sides perpendicular to the Bases; and from the greater, CF , take EF , equal to AB . The Solid BM is to the Solid EK , *per* 32. as AM is to FK ; that is, by the Hypothesis, as AB is to AB ; that is, by the Construction, as CF is to EF ; that is, as CG is to $(a) EG$; that is, (b) as the (a) *Per* 1. and CK is to the same Solid EK . Therefore the Solids BM and CK have the same Proportion to EK : Therefore BM and CK are equal. *Q. E. D.* *Per* 25. l. 11.

Corollaries.

WHAT Affections have been demonstrated of Paralleloepipeds, *Prop.* 29, 30, 31, 32, 33, 34, do also hold to Triangular Prisms, which are the Halves of Paralleloepipeds. As is manifest from *Prop.* 28. Therefore, The Triangular Prisms, which are of equal Height, *Fig.* 37. as their Bases AB .

If they be like, their Proportion is triplicate to the Portion of the Sides, opposite to the Angles.

If they be equal, they reciprocate their Bases and Altitudes; and if they reciprocate their Bases and Altitudes, they are equal.

Scholium.

WHAT hath here, in *Prop.* 34. been shew'd of Paralleloepipeds, will be demonstrated in the 12th Book of *Elements*, *Prop.* 9. Of all Prisms whatsoever, *Corollary* 3. *Prop.* 9. Of Cones and Cylinders, *Prop.* 15.

PROP. XXXV.

very long, and subservient to the following Proposition, which we will demonstrate without

PROP. XXXVI. Theorem.

Fig. 38. **A** Paralleloepid (*DH*) made of three proportional right Lines (*A, B, C,*) is equal to Paralleloepid (*IN,*) which is made of the Mean (*B,*) and is Equiangular to the former.

Let the Base *FD*, of the Paralleloepid *DH*, have Side *EF* equal to *A*, and the other Side *ED* equal to *C*. And the Side *EG*, which stands upon the Base equal to *B*. Thus the Paralleloepid *DH* will be made of the three right Lines, *A, B, C*. Then let the three Sides, *LX, XM*, and consequently all the rest, of the Paralleloepid *IN* be equal to the middle Line *B*: And the solid Angle *X*, equal to the solid Angle *E*; the Paralleloepid will be made of the Mean *B*, and be Equiangular to the former. I say also that it is equal.

For, seeing, by the Hypothesis and the Construction *FE* is to *LX*, so reciprocally, *IX* is to *DE*, the Bases

(a) *Per* 14. (a) *DF, IL*, will be equal. Now, because the

l. 6.

(b) *Per* another, (b) they will coincide; and because of the

Defn. 11.

l. 11.

Equality of the right Lines, *EG, XM*, the Points *M* and *G* will coincide. Wherefore both the Solids will have perpendicular Altitude; to wit, the right Line, which

let fall from the Points *M, G*, now become one, unto the

* *Per* 31.

l. 11.

Plane of the Base. The Solids therefore *DH, IN*, are equal. *Q. E. D.*

Scholium.

WE will further observe what is of great Use, that three Lines drawn into or multiplied one by another after what manner soever, a Solid of the same Magnitude is produced.

ABC. CAB. BCA.

1.

2.

3.

In the present Scheme, the two first Letters designate the Base; the third, the Altitude. Let us compare the first with the second.

The Base *AB* is to the Base *CA*, *per* 1. *l. 6.* as the Side *B* is to the Side *C*; that is, reciprocally, as the Height *B* is to the Height *C*. Therefore, by *Prop. 34.*

ABC is equal to CAB .

In the same manner it may be shew'd that the first is equal to the third, and the third to the second.

PROP. XXXVII. Theorem.

Parallelepipeds which are like, and described in the like manner by proportional right Lines, will themselves also be proportional; and conversely.

This is manifest of itself. For the Proportions of Parallelepipeds, by the 33d of this Book, will be triplicate to those Proportions which, by the Hypothesis, are equal, which the Lines have betwixt themselves.

The Converse is manifest of itself also.

The Proposition is true of all sorts of like Bodies, which will appear from Book the 12th, to have betwixt themselves a Proportion triplicate to that which the Sides have.

PROP. XXXVIII, XXXIX.

THESE contain nothing remarkable, and are scarce of any Use.

PROP. XL. Theorem.

THIS is of a small Use, and indeed no other than the 28th Proposition in another View.

Scholium.

FROM what hath hitherto been demonstrated, we have the Dimension of Triangular Prisms, and of Quadrangular or Parallelepipeds; to wit, if the Altitude be multiplied into the Base. As if the Altitude be of 10 Feet, and the Base of 100 square Feet, (now the Base is measured by *Scholium*, p. 36, or 41. l. 1.) multiply 10 by 100, there will arise 1000 Cubic Feet for the Solidity of the given Prism.

The Demonstration is easy. For, like as a Rectang
ariseeth from the Multiplication of one Side by another
so a right Paralleloepid is produced from the Height
drawn into the Base. Therefore every Paralleloepid is al
produced from the Altitude multiply'd into the Base; se
ing by 31. *l.* 11. it is equal to a right Paralleloepid, co
stituted upon the same Base with the same Height.

Then seeing the whole Paralleloepid is produced from
the Height into the whole Base; the half of the Paralle
pepid (that is, a Triangular Prism by 28. *l.* 11.) will
be produced from the Altitude multiplied by half the Base
to wit, the Triangle I L K.

Fig. 29.



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T H E

Elements of EUCLID.

B O O K XII.

With Us the Eighth.

WHAT in the foregoing Books we have endeavour'd to perform; namely, to bring the Elements of the Mathematicks into a more easy and brief Method, shall be to be endeavour'd in this Twelfth Book especially; the Doctrine whereof is most necessary, but the Demonstrations are so prolix, that they commonly make Beginners almost to despair. We have so propos'd to ourselves to remedy this Evil, that in the mean while we will not part from the Rigour of Geometrical Demonstration. Which Thing, whether or no we have attain'd, the Reader will understand, if he shall compare this of ours with *Euclid's* Prolixity.

Now, after *Euclid* had in the former Book declared the Elements of Solids, and defined the Measures of the most easy Bodies, those, namely, which are terminated with plain Surfaces; In this Twelfth Book he considers Bodies bounded with curve Surfaces; to wit, Cylinders, Cones and Spheres; compares them betwixt themselves; and defines their Measures. This Book is indeed most profitable, because it contains those principles on which the chief Masters of Geometry, and especially *Archimedes*, have built so many famous Demonstrations, concerning the Cylinder, Cone and Sphere.

DE-

DEFINITIONS.

Fig. 1. 12. 1. A Pyramid is a Solid [Z L] comprehended under the Triangles [A L C], [C L F, F L B, B L A] placed from one Plane [Z] to one Point [L]

The Plane Z is called the Base, and may be either Triangle or Quadrangle, or any other Figure, from each of the Sides whereof there arise Triangles meeting together at the Point L, which is called the Vertex, or Top.

As the Triangle amongst Rectilinear plane Figures, the Pyramid amongst solid ones is the first and most simple.

Fig. 2, 3. 2. If without the Plane of some Circle [C L] there be taken the Point [A,] and from it be drawn the infinite right Line [A F,] touching the Circle in C; and this Line (the Point [A] remaining fix'd) be turn'd about the Circumference of the Circle, until it returns thither, from whence it began to be moved; the Surface described by the right Line [A C F] is term'd a conical Surface; and the Body which is contain'd under this Surface, and the Circle [C L] is call'd a Cone.

The Vertex of the Cone is [A.]

The Circle [C L] is the Base of the Cone.

The right Line [A B,] drawn from the Vertex to the Center of the Base, is the Axis of the Cone.

The Side of the Cone is the right Line [A C] drawn from the Vertex to the Circumference of the Base, which that it is wholly in the Surface of the Cone, is manifest from the Production of the Figure.

* *Fig. 2.* A right * Cone is, when the Axis [A B] is perpendicular to the Base.

† *Fig. 3.* A scalene † or oblique Cone, is, when the Axis [A B] is not perpendicular to the Base.

A right Cone is also made by a right-angled Triangle [C B A] turn'd round about one of the perpendicular Sides [A B.] See *Fig. 2.*

Fig. 4, 5. 3. If an infinite right Line [C O F] be turn'd about two Circles [C L, O Q] equal and parallel, until it returns to that Place from whence it began to be mov'd, and remains always, whilst it is mov'd, parallel to it self, the Surface described by the right Line [C O F] is called a Cylindrical Surface; and the Body which is contain'd under this Surface, and the two Circles, is call'd a Cylinder.

The Bases of the Cylinder are the Circles [CL, OQ] the right Line [A B] which connects the Centers of the Bases, is called the Axis. The right Line [O C] in the Surface of the Cylinder, touching both the Bases, is called Side of the Cylinder.

A right Cylinder, is when the Axis is perpendicular to *Fig. 4.* the Base.

A scalene or oblique Cylinder, is, when the Axis is not *Fig. 5.* perpendicular to the Base.

A right Cylinder, is also made by a Rectangle [O C B A] turned round about one Side [B A.] See *Fig. 4.*

4. Like Cones and Cylinders are those, which have their Axis [A K, Z O] and the Diameters of their Bases [B F, Q R] proportional. *Fig. 20, 21.*

5. A Sphere, is a Solid contained under one Surface, unto which Surface all the right Lines that are drawn from a certain Point within the Figure, are equal amongst themselves. That Point is called the Center. The Diameter of the Sphere is a right Line drawn through the Center unto the Surface on both Sides.

A Sphere is produced if a Semi-circle be turned about *Fig. 6.* its Diameter [A F] which remains in the mean while unmoved.

6. Magnitudes inscribed into, or described about, some Figure, whether they be greater or lesser than the Figure, are then said to *end* in the Figure, when they will at the last differ from it by a Quantity less than any given one whatsoever, or how small soever.

Therefore if those Magnitudes which are inscribed into some Figure, will at last fall short of it by a Deficiency less than any given one whatsoever, the Magnitudes inscribed are said to *end* in the Figure; and if those which are circumscribed about some Figure, will at last exceed it by an Excess less than any given one whatsoever, they shall be said to *end* in the Figure.

PROPOSITION I. Theorem.

Fig. 6, 7. **THE** Proportion of like Polygons inscribed in a Circle, is duplicate to the Proportion of the Diameters (AF , IC .)

Let AO , BF ; IR , LC , be drawn. Because the Polygons are supposed to be like, the Angles (OBA , RLI) will (*per Defn. 1. l. 6.*) be equal; and the Sides OB , BA proportional to the Sides RL , LI . Therefore in the Triangles AOB , RLI (*per 6. l. 6.*) the Angles O and R are equal. Therefore also the Angles BFA and LCI , which stand upon the same Arches, BA , LI , are (*per 21. l. 3.*) equal. But the Angles, FBA , CLI , in Semi-circles, are (*per 31. l. 3.*) right ones. Therefore the other Angle (*per Corol. 9. p. 32. l. 1.*) BAF , LIC , are equal. Therefore because the Triangles FAB , CIL , are Equiangular to each other, they are (*p. 4. l. 6.*) like: and BA will be to LI , as AF to IC . Now, because, by the Hypothesis the Polygons are like, their Proportion will be duplicate (*p. 20. l. 6.*) to the Proportion of the Sides BA , LI ; that is, as I have already shewed, duplicate to the Proportion of the Diameters AF , IC . *Q. E. D.*

Corollary:

Fig. 6, 7. **THE** Circumferences of like Polygons inscribed in a Circle are betwixt themselves as the Diameters. Seeing it hath already been shewed, that AB is to LI , as AF is to IC ; OB will also be to RL , as AF to IC . And so of the rest of the Sides. Therefore all the Sides together, will be to all the Sides together, that is, one Circumference to another, as AF is to IC .

A Lemma.

Fig. 8. **P**OLYGONS inscribed in a Circle, end in a Circle. Is scribe a Square, as $ACBD$. Seeing this is half (*p. Schol. p. 6. and 7. l. 4.*) of the Square which is circumscribed to the Circle, it will be greater than half of the Circle. Wherefore

this be taken out of the Circle, there will be taken out of more than half. Then each Arch being bisected in E, X, I, H, inscribe an Octagon: and let FG touch the Circle in E; which FG, let BC, DA meet in G and F; CF will be a Parallelogram, of which, seeing the Triangle CEA (*per* 41. l. 1.) is half, this will be more than half of the Segment CEA. In the same manner each of the Triangles, AKD, DIB, &c. is more than half each of the Segments. Therefore all the Triangles are more than half all the Segments. Therefore if you take these out of those, that is, out of the Remainder of the Circle, more than half will be taken away. In the same way of arguing, if there be inscribed in the Circle, Polygons of Sides always double in Number; I can shew that there will always be taken out of the Remainder of the Circle more than half. Therefore the Remainder must at last be less than any given one whatsoever; and consequently the inscribed Polygons will at last fall short of a Circle by a Quantity less than any given one whatsoever; that is, (*per* Defn. 6. l. 12.) will end in a Circle.

PROP. II. Theorem.

THE Proportion of Circles is duplicate to the Fig. 6, 7.
Proportion of their Diameters.

The Proportion of Polygons inscribed in a Circle without End, is (*per* 1. l. 12.) duplicate to the Proportion of the Diameters. But Polygons (by the foregoing Lemma) inscribed in a Circle infinitely, at last end in the Circle. Therefore the Proportion of Circles is also duplicate to the proportion of the Diameters.

PROP. III, IV.

ARE prolix, and hard for young Beginners, and have no other Use, than that they serve to the Demonstration of the Fifth, which we shall demonstrate much more easily without them.

Lemmata,

*Lemmata, or preparatory Propositions to Prop. V**Lemma I.*

Fig. 9.

IF two Triangular Pyramids be cut with Planes (O S R X Z) parallel to the Bases (A B C, I Q V) which same Planes divide the Sides (C F, Q L) proportionally in (E and Z, then O S E, R X Z will be betwixt themselves, as the Bases (A C B, I Q V.)

Because the parallel Planes, O S E, A B C, are cut by the Planes B F C, A F B, A F C, the common Sections, S E, B C, and O S, A B, and O E, A C, (will be *per 16. l. 11*) parallel. Wherefore the Angles O S E, A B C, and S O E, B A C, and O E S, A C B, two and two, are (*per 10. l. 11*) equal. Wherefore the Sections, O S E, A B C, are like (*per 4. l. 6*) In the same manner I might shew that the Sections R X Z, I V Q, are like. Therefore (*per 19. l. 6*) the Proportion of the Section A B C, to the Section O S E is duplicate to the Proportion of the Side B C, to the Side S E; and the Proportion of the Section I V Q to R X Z, duplicate to the Proportion of V Q to X Z. But the Proportions of B C to S E, and of V Q to X Z, are the same for B C is to S E (by *Corollary 1. per 4. l. 6.*) as C F to E F; that is, by the Hypothesis, as Q L to Z L; that is (by the same *Coroll.*) as V Q to X Z. Therefore the Proportion of A B C to O S E, is the same with the Proportion I V Q to R X Z. *Q. E. D.*

Lemma II.

Fig. 10.

PRISMS inscribed infinitely in a Pyramid (Z C A B) which hath a Triangular Base, end in the same Pyramid

Let the Side of the Pyramid be divided into a certain Number of equal Parts, A B, B G, G F, and through A and G, there being made the Sections, G D N and B E M parallel to the Base Z A C; let the Triangular Prisms B E P M A O and G D N K B Q be understood to be inscribed in the Pyramid. These then being continued without the Pyramid, let there be understood to be described about the Pyramid the Prisms C I B A, P X G B, N H F G. The Excesses of the circumscribed Prisms above the inscribed ones, are the Solids I M, X K, H G, which, taken together

er, are equal to the Prism C I B A: For H G (*per* 25. 11.) is equal to D B, and consequently H G with X K, are equal to P X G B, that is, (by the same) to M E B A. Therefore the three, H G, X K, I M, are equal to the Whole, C I B A. But if A F be divided without End into more equal Parts, and consequently the Number of Prisms be infinitely increased, A B will become less than any given line. Therefore (as it is manifest from *p. 25. l. 11.*) the prism C I B A will become less than any given one. Therefore the Excess of their circumscribed Prisms, (and much more of the Pyramid Z C A F, which is part of the Prisms circumscribed about it) above the inscribed Prisms, will be less than any given Prism. Therefore the inscribed Prisms (*by Defn. 6. l. 12.*) end at last in a Pyramid. Q. E. D.

PROP. V: Theorem:

Triangular Pyramids of the same Height have that Proportion betwixt themselves, which their Bases (*A Q R, E S X*) have.

Let the equal Altitudes of the Pyramids be represented *Fig. 11.* by the Side A P, E Z; which, on both Sides, let be divided into as many equal Parts as you will, but so that they be of the same Number; and let there be made through the Points of the Divisions, Sections parallel to the Bases: Let Triangular Prisms, of the same Number, and the same Height, be understood to be inscribed in both Pyramids. And now because the Prisms, L A, I E, are of the same Height, the Prism L A will be to the Prism I E (by *Coroll. 1. p. 34. l. 11.*) as the Base L O B is to the Base I N K; that is, by (*Lemma 1.*) as the Base Q R A is to the Base S X E. In the same manner I might shew that each of the Prisms inscribed into the Pyramid Q P A R, is to each inscribed into the Pyramid S Z E X, as the Base Q A R is to the Base S E X. Therefore all of them together, are to all of them together, as Base is to Base. Wherefore seeing they at last end (*per Lem. 2.*) in the Pyramids themselves, the Pyramids themselves also will be as their Bases. Q. E. D.

PROP.

PROP. VI. Theorem.

Fig. 12, 13. **A**LL Pyramids whatsoever, which are of equal Height, have that Proportion betwixt themselves, which their Bases (AB , CFO) have:

Let their Bases be resolved into Triangles, A , B , C , O : and the whole Pyramids into Triangular Pyramids. The Pyramid AX , is to the Pyramid OZ (by the foregoing) as A is to O ; and the Pyramid BX , is to the Pyramid OZ , as B is to O (by the same.) Therefore the Pyramids AX , BX , together (that is, the whole Pyramid ABX) are to the Pyramid OZ , as A , B , together, are to O . By the same Argumentation, the Pyramid ABX , is to the Pyramid FZ (by the foregoing) as AB , are to F . And ABX , is to CZ , as AB , is to C . Therefore ABX is to the three, OZ , FZ , CZ , together; that is, to the whole Pyramid, OF , CZ , as, AB , together, is to O , F , together. *Q. E. D.*

PROP. VII. Theorem.

Fig. 14. **E**VERY Pyramid is the third Part of a Prism which hath the same Base and Height.

First, let the Triangular Pyramid, $BGAC$, have the same Base and Height with the Prism, $BACFEO$: Let BF , AO , AF , be drawn. The Triangles, BFC , BOA , are (per 34. l. 1.) equal. Therefore the Pyramid, BFC , is equal to the Pyramid, BOA . For the same Reason, OEA , is equal to the Pyramid, OBA ; that is, the Pyramid, BOA , for they are the same Pyramid. Therefore BFC , and OEA , are also equal. Therefore all three, BFC , OEA , OBA , or BOA , are equal. Therefore the three together are triple of one BFC . But those three constitute the Prism, $BACFEO$. That Prism therefore is triple to the Pyramid, BFC , that is, (per 5. l. 11.) to $BGAC$. *Q. E. D.*

Fig. 15. Then let any Pyramid whatsoever have the same Base and Height with the Prism, $AEFH$: The Lines BO , BE , and NI , NG , NH , being drawn, resolve the Prism into Triangular Prisms, and the Pyramid into Tri-

angular

angular Pyramids. Which being done, the Demonstration manifest from the first Part: For each Part of the Prisms will be triple of each Part of the Pyramids. And consequently the whole Prism will be triple to the whole Pyramid. *Q. E. D.*

PROP. VIII. Theorem.

THE Proportion of like Pyramids ($OACB$, $KHIN$) is triplicate to that which the homologous Sides (AB , HN) have to each other.

First let them be Triangular: The Parallelograms, AM Fig. 16. and HQ being perfected, set upon them the Paralleloepids, AG , HL , in the Height of the Pyramids; which seeing the Pyramids are like, will also, (as appears from *Defn.* 9. 11.) be like. Then let EF , RP , be drawn; and through EF , CB , as likewise through RP , IN , the Paralleloepid will be cut (*per* 28. l. 11.) into two equal Prisms: each of which will be triple to the Pyramids, $OACB$ and $KHIN$ (by the foregoing). Therefore both together, that is, the whole Paralleloepids, AG , HL , will be six-fold the Pyramids. Therefore the Pyramids are proportional to the Paralleloepids. But (*per* 33. l. 11.) the Proportion of these each to other is triplicate to the Proportion of the Sides, AB , HN . Therefore so likewise is the Proportion of the Pyramids.

But if the like Pyramids shall be polygonal, let them be Fig. 17. solved into the Triangular ones, AR , BR , CR and OK , EK , FK . You may from 20. and 5. l. 6. and *Defn.* 9. 11. easily shew, that AR is like to OK , and BR to EK , and CR to FK . Therefore, by the former Part, the Proportion of the Pyramids, AR , OK , is triplicate to the Proportion of IM to PZ : And the Proportion of the Pyramids, BR and EK , is triplicate to the Proportion of MX to SZ ; that is, again, by the Hypothesis, of IM to PZ ; and the Proportion of the Pyramids, CR , FK , is triplicate to the Proportion of XQ to ST ; that is, again, of IM to PZ . Seeing therefore the Proportion of each to IM is triplicate to the Proportion of IM to PZ , the Proportion also of all to all (that is, the Proportion of the whole Pyramid, $ABCR$, to the whole, $OEFK$) will be triplicate to the Proportion of IM to PZ . *Q. E. D.*

PROP.

PROP. IX. Theorem.

Fig. 18, 19. **E***QUAL* Pyramids have their Bases and Altitudes reciprocally proportional; and those which have them so, are equal.

Part I. First, let the Pyramids be Triangular, BAC and KHN : The Parallelograms BE , HR , being perfected upon these set the Paralleloepids, BF , HP . These will be (as was shewed in the foregoing) six-fold of Pyramids which are, by the Hypothesis, equal, and consequently will be equal betwixt themselves. But now the Altitudes of these Paralleloepids HK , BA , are the same with those of the Pyramids; and the Bases BE , HR , are double to the pyramidal Bases, (*per 34. l. 11.*) BCO , HNL , and consequently proportional to them. Seeing therefore by reason of the Equality of the Paralleloepids, as BE is to HR so (by the same) is reciprocally HK , to BA ; it will also be that, as the Base BCO is to the Base HNL , so, reciprocally, is the Altitude HK to the Altitude BA . *Q. E. D.*

But if the Pyramids have polygonal Bases, let them be reduced into Triangular ones, retaining the same Altitude and these will be equal to those by the sixth. But the Pyramids thus reduced have, as we have now demonstrated, their Bases and Altitudes reciprocally proportional. Therefore the given polygonal Pyramids also have their Bases and Altitudes reciprocally proportional. *Q. E. D.*

Part II. Because it is now supposed, that BCO is to HNL , as HK is to BA ; BE will also be to HR , as HK is to BA . Therefore the Paralleloepids, BF , HP , are (*per 34. l. 11.*) also equal. Therefore their sixth Parts also, to wit, the Pyramids BAC , KHN , are equal. *Q. E. D.*

Corollaries.

WHAT has been demonstrated of Pyramids in Proposition 6, 8, 9. does also agree to all Prisms whatsoever seeing these are (*per 7. l. 12.*) triple to Pyramids which have the same Bases and Altitudes. Therefore,

1. In Prisms of the same Height, their Proportion is the same as that of their Bases. For this was shewed of Pyramids, *Prop. 6.*

2. T

The Proportion of like Prisms is triplicate to the Proportion of their homologous Sides. For this was shew'd concerning Pyramids, *Prop. 8*.

Equal Prisms have their Bases and Altitudes reciprocally proportional; and those which have them so are equal. This is shew'd of Pyramids, *Prop. 9*.

It is strange that these Things were pass'd over by Euclid, seeing they are the chief Things which can be deliver'd concerning Rectilinear Solids.

Scholium.

FROM what has been hitherto demonstrated is drawn the Method of measuring any Prisms or Pyramids whatsoever.

The Solidity of a Prism is produced from the Altitude multiplied into the Base; and that of a Pyramid from the third Part of the Altitude multiplied by the Base.

As if the Altitude of a Prism be of 5 Feet, but the Base contains 25 Square Feet; multiply 25 by 5, and there arise cubic Feet for the Solidity of the Prism.

For let there be a polygonal Prism, as A H. And let *Fig. 15, 16.*

Triangle B A C be understood to be equal to its Base

and upon B A C, the Prism B E to be set at equal

Height with A H. The Prisms B E, A H will be (by

Corollary 1. foregoing) equal. But the Prism B E (by

Prop. 40. l. 11.) is produced from its Altitude drawn into

Base B A C; that is, into A E, by Construction.

Therefore the Prism A H also is made of its Base A E,

multiplied by its Height, which is supposed to be equal

to the Height of the Prism B E.

From hence, and from the 7th, the Demonstration of

the second Part is also manifest.

A Lemma to Proposition 10.

PYRAMIDS and Prisms, which are inscrib'd in Cones and Cylinders infinitely, do at last end in the Cones and Cylinders.

This is demonstrated as the *Lemma* of *Proposition 2.* with help of *Proposition 6.* and of *Corollary 1.* after *Proposition 1.* as their Planes inscrib'd in a Circle, so here Prisms and Pyramids which stand upon those Planes as their Bases, be continually taken away from the Cones and Cylinders.

N

P R O P.

PROP. X. Theorem.

Fig. 20. **EVERY** Cone is a third Part of a Cylinder having the same Base and Height.

Let a regular Polygon of as many Sides as you please be understood to be inscrib'd in the Base CL , and upon it the Base, for a Cone let a Pyramid, and for a Cylinder a Prism be inscrib'd. The Pyramid (*per 7. l. 12.*) will be a third Part of the Prism. And if again in the Circle a Polygon of twice as many Sides be inscrib'd, and upon it be inscrib'd for a Cone a Pyramid, but for the Cylinder a Prism, the Pyramid will again be a third Part of the Prism. Thus it will always be. Wherefore seeing Pyramids are to a Cone, and Prisms to a Cylinder, the Cone also will be a third Part of the Cylinder. *Q. E. D.*

PROP. XI. Theorem.

Fig. 20, 21. **CONES** of equal Height (BAF , QXR) as their Bases (CL , SE .) The same Thing belongs to Cylinders of equal Height also.

Pyramids inscrib'd into Cones of equal Height, as their Bases, (*per 6. l. 12.*) But Pyramids do at length fill up the Cones. Therefore Cones also are as their Bases. Seeing Cylinders are three-fold of Cones, which have the same Base and Altitude with them, they also will be three-fold of their Bases. *Q. E. D.*

Corollary.

IN the same manner it may be demonstrated, that Prisms and Cylinders of equal Height are betwixt themselves as their Bases; yea, that all cylindrical Bodies of the same Altitude; that is, which are produced from whatever Planes multiplied by the same Altitude, are betwixt themselves as their Bases. You may reason in the same manner of Pyramids and Cones of equal Altitude, and of all conical Bodies whatsoever.

PROP. XII. Theorem.

THE Proportion of like Cones (BAF and QZR) is triplicate to the Proportion of the Diameters (BF and QR) which are in the Bases. The same thing is to be said of like Cylinders.

Upon the Bases of the like Cones, let regular Polygons be described, which Polygons consequently will be like. The Pyramids which are inscribed upon these Polygons will also be like; as may be easily shew'd. Therefore their Proportion is triplicate (*per* 8. l. 12) to the Proportion of the Diameters BF , QR ; that is, to the Proportion of the Diameters BF , QR . Wherefore seeing the Pyramids end in the same Point, the Proportion also of the Cones is triplicate to the Proportion of the Diameters BF , QR . $Q.E.D.$
 The Theorem is manifest of Cylinders, seeing they are composed of Cones.

PROP. XIII. Theorem.

If a Cylinder (BI) be cut with a Plane (RL , Fig. 22.) parallel to the Bases (BQ , CI ;) one Part, (BL ;) shall be to the other Part, (RI ;) as one Segment of the Axis (AO) is to the other Segment of the Axis (OF .)

This Proposition is demonstrated, as at the first of l. 6. The Theorem is in the same manner true of the Super-

PROP. XIV. Theorem.

Cylinders (AR and CI) of equal Bases (MQ , GH) are as their Altitudes (LZ , ST ;) The same thing happens to Cones.

If we cut off from the higher Cylinder AR , the Cylinder whose Height LE is the same with SF . Therefore (*per* 1. l. 12.) the Cylinders AO , CI , are equal. Seeing therefore the Cylinder AO , is to the Cylinder AR , (by the

P R N 2 the

the foregoing) as LE is to LZ ; CI also shall be to LE as LE is to LZ ; that is, (because LE and SF are equal by Construction) as SF to LZ . *Q. E. D.*

Corollary.

THE Theorem is also true of Prisms, and likewise of Pyramids, and the Demonstration altogether alike of Prisms, the thing is demonstrated from *Corol. 1. p. 12.* and *25. l. 11.* and its *Corol.* Of Pyramids from *and from p. 7. l. 12.*

PROP. XV. Theorem.

Fig. 24, 25. **E***QUAL* *Cylinders* (AR , DF) *have their Bases and Altitudes reciprocally proportional; and if they have them so, they are equal. The same is true of Cones.*

This is demonstrated, as *Prop. 34. l. 11.* only for *32. 25. l. 11.* there cited, there must be cited here *Prop. 11. and 13. l. 12.*

Scholium.

Fig. 25, 24. **W**HEREAS *Euclid* hath said nothing of compound Proportion in Bodies, we shall briefly demonstrate this Place.

1. A Cylinder hath to a Cylinder, and a Prism to a Prism, a Proportion compounded of the Proportions of Bases and Altitudes.

Let FD and AR be Cylinders of different Altitudes in those of equal Altitude the Thing is manifest.) the higher, cut off AO of equal Height with FD . let the Proportion be thus; as the Base UT is to the Base MQ , so FN to X ; and as the Altitude ND is to the Altitude BR , so is X to Z . We must then shew, that the Cylinder FD is to the Cylinder AR as FN is to Z . The Cylinder FD is to the Cylinder AO (per *11. l. 12*) as the Base VT is to the Base MQ ; is, (by Construction) as FN is to X ; but the Cylinder AO is to the Cylinder AR (per *13. l. 12.*) as BO to BR .

by Construction) as X to Z. Therefore, by Proportion of Equality, the Cylinder F D is to the Cylinder A R, N to Z.

The Proposition may be demonstrated in the same manner of Prisms, but from *Cor. 1. p. 9.* and *Cor. p. 14.*

A Cone hath also to a Cone, and a Pyramid to a Pyramid, a Proportion which is compounded of the Proportion of Base to Base, and Altitude to Altitude.

(by *Prop. 10.* and *7. l. 12.*) they are third Parts of Cylinders and Prisms.

PROP. XVI, XVII.

THESE Propositions, the most prolix of all the others, have no other Use than to serve to the demonstrating, Prop. 18. we shall demonstrate in a more easy Way.

Lemma to Proposition 18.

CYLINDERS inscrib'd in an Hemisphere end in the *Fig. 26.* Hemisphere. Let P Z Y be the greatest Semi-circle of the Hemisphere; and let the Radius A Z be perpendicular to the Diameter P Y. Cut A Z into a certain Number of Parts, A M, M N, N Z; and there being drawn Lines through the Points of the Divisions M, N, the perpendiculars B O, &c. Let there be inscrib'd in the Semi-circle the Rectangles O B R K, E D H S; which after being continued without the Semi-circle, let there be understood to be describ'd about the Semi-circle the Rectangles F T Y P, L V B O, Q X D E. They will all of them be of the same Height, and the Excesses of the Circumscribed ones, above those which are inscrib'd, will be equal to the Rectangles F K, L S, X E, V H, T R, which, taken together, make the Rectangle F T Y P. For because X E is equal to D S, those L S, V H, X E, together, will be equal to the Rectangle L B, that is, O R. Wherefore if we add on both Sides the Planes F K, T R, all those, F K, L S, X E, V H, T R, taken together, will be equal to the Rectangle F T Y P. If now the Semi-circle, with the Rectangles, be understood to be turn'd about the Radius A Z, the Cylinder is in the mean while unmov'd, the inscribed Rectangles

angles EH , OR , will produce Cylinders inscribed in Hemisphere; and the circumscribed Rectangles will produce Cylinders circumscribed about the Hemisphere, fitting one upon another; and as the Excesses of the circumscribed Rectangles above the inscribed ones, was the angle FY ; so likewise the Excesses of the circumscribed Cylinders above the inscribed ones, will be the Cylinder which is produced from the Rectangle FY . But now Altitude of this Cylinder will be made less than any given Height; and consequently (as is manifest from 13. 1.) itself will grow to be less than any given Cylinder, its Radius being divided into more equal Parts without the Number of Rectangles, and from thence of Cylinders be infinitely encreased. Therefore the Excess of the circumscribed Cylinders, and much more of the Hemisphere itself, which is only a Part of the Circumscribed Cylinder above the inscribed ones, will at last become less than any given one. Therefore (by *Defn.* 6. l. 12.) Cylinders infinitely inscribed in an Hemisphere, do at length end in the Hemisphere itself. *Q. E. D.*

Corollary.

IN the same manner it will be demonstrated, that Cylinders inscribed in a Cone, Conoid, Spheroid, &c. do at last end in the same.

P R O P. XVIII. Theorem.

Fig. 27. **T**HE Proportion of Spheres is triplicate to the triplicate Proportion of their Diameters (BK , RZ .)

The Radiuses AB , YR , being divided into as many equal Parts as you will, but of an equal Number, and then Lines drawn through the Points of the Divisions perpendicular, &c. Let Rectangles of an equal Number be understood to be inscribed in the greater Semi-circles of the Spheres, which Rectangle being turn'd about, the unequal Radiuses AB , YR , will be conceiv'd to inscribe in both Hemispheres, a like Number of Cylinders standing one upon another. Now, because KC is (per *Cor.* p. 3. l. 1.) as CF , as CF is to CB ; the Proportion of KC to C

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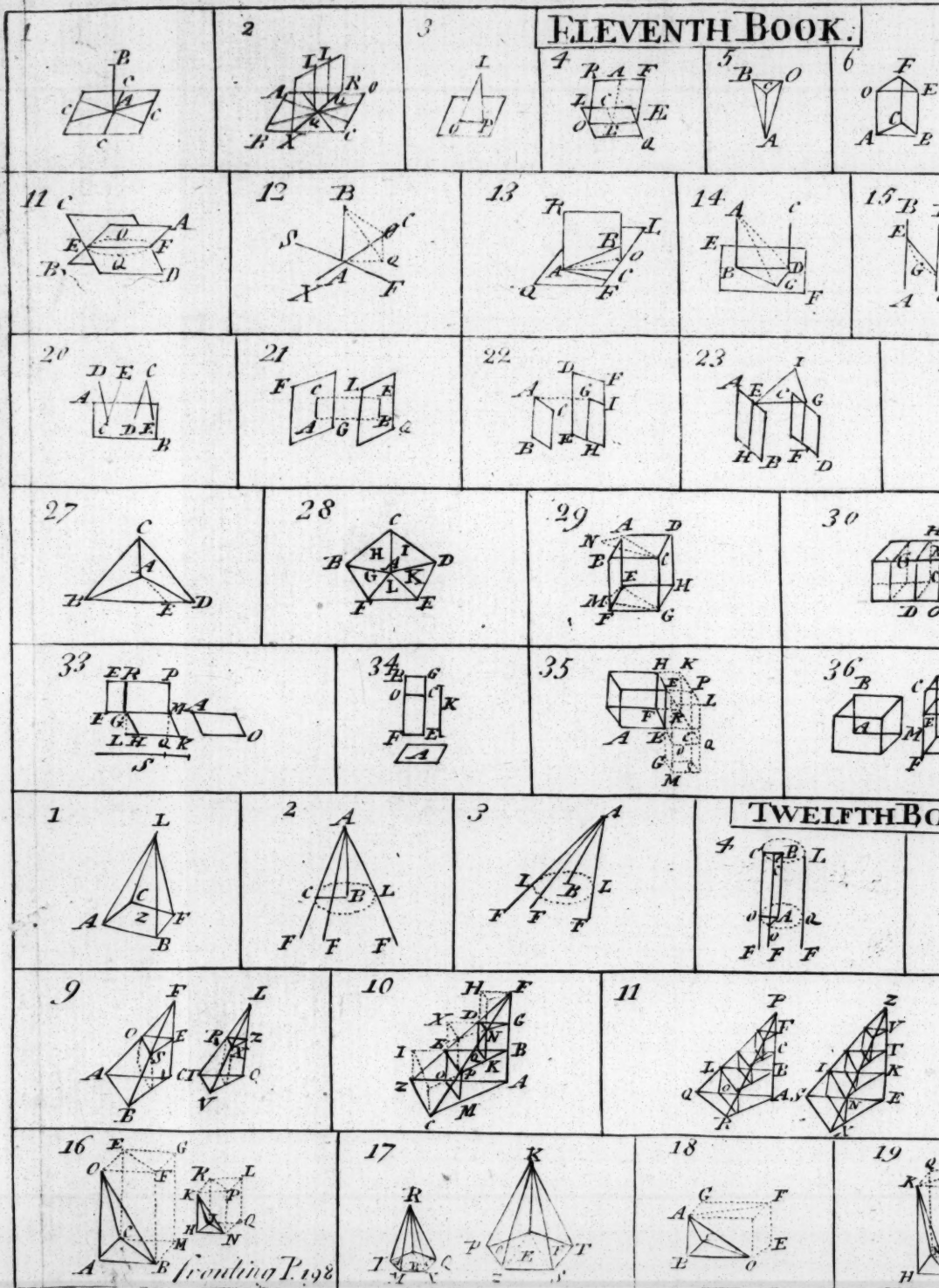
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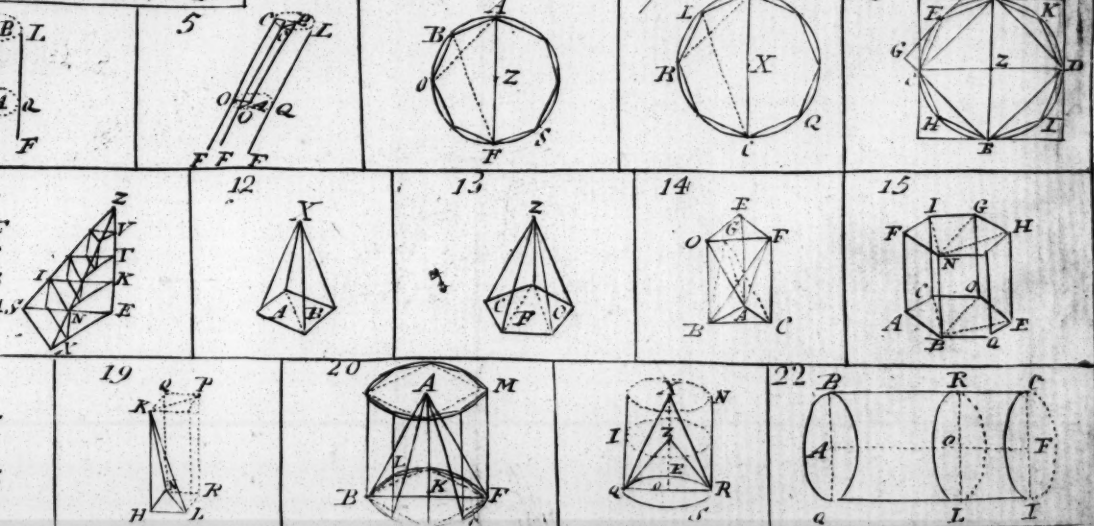
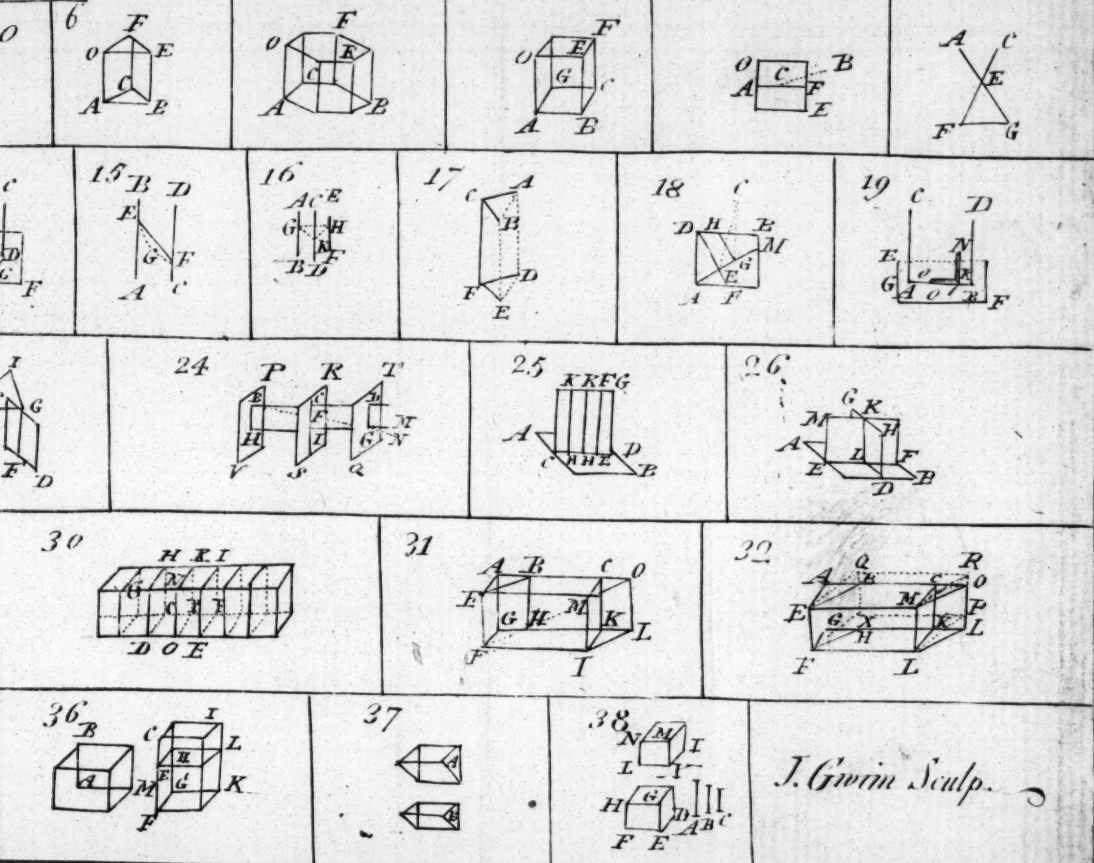
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ib. XII.

Defn. 10. l. 5.
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Defin. 10. *l.* 5.) will be duplicate to that of KC to CF , that is, to the Proportion of FC to CB . In like manner the Proportion of ZE to ER , will be duplicate to the Proportion of XE to ER . But, by the Construction, KC is to CB , as ZE is to ER . Therefore FC also is to CB , as ZE to ER . But BC , by the Construction, is to CO , as ZE to ES . Therefore by Equality, FC is to CO , as ZE is to ES . Therefore (by *Defin.* 4. *l.* 12.) the Cylinders FL , XQ , are like, and consequently their Proportion is (per 12. *l.* 12.) triplicate to the Proportion of their Diameters, FL , XV , or of the Semi-diameters, FC , CE , which are the Bases. But the Proportion of FC to CE , is the same with the Proportion which is betwixt the Diameters of the Spheres BK , RZ ; for, as I have already shew'd, FC is to XE , as CO is to ES ; that is, as BK is to RZ , which, by the Construction, are Equimultiples of those, CO , ES .) Therefore the Proportion of the Cylinders, FL , XQ , is triplicate to the Proportion of the Diameters, BK , RZ . In the same manner we might demonstrate that each Cylinder inscribed in one Hemisphere, bears to each Cylinder inscribed in the other Hemisphere, a Proportion triplicate to the Proportion of the Diameters, BK , RZ . Therefore also the Proportion of all together, all together, is triplicate to the Proportion of the Diameters BK , RZ . Wherefore seeing the Aggregates of the Cylinders do at length end in their Hemispheres, the Proportion of the Hemispheres also, and consequently of the Spheres, will be triplicate to the Proportion of their Diameters. *Q. E. D.*

Corollary.

THEREFORE the Proportion of the Diameters being known, the Proportion of the Spheres becomes known likewise. As if the Diameter of the lesser be one Foot, that of the greater ten Feet; let the Proportion of one to ten be continued through four Terms, 1, 10, 100, 1000; then 1, the first is to 1000, the fourth Term, so is the lesser Sphere to the greater.

The Dimension of Cones, Cylinders and of the Sphere, will be exhibited in the following Book out of *Archimedes*.

Scholium.

Scholium.

AS like plain Figures are encreas'd or diminish'd in a given Proportion by one mean Proportional, so like Bodies are encreased or diminished by two mean Proportionals.

Let a Sphere, or Cube, or any other Body whatsoever be given, whose Radius, or Side, is A. Likewise let a Proportion whatsoever of A to B be given, as the double or 2 to 1. A Body is to be discovered both double to the given one, and like to it.

Betwixt the Terms of the given Proportion A and B let there be found two mean Proportionals X, Z, according to what was taught in the *Scholium* of *Prop. 13. l. 6.* A Sphere whose Radius is X, or other Body like to the given one, which is made upon the Side X, will be double to the given one.

For like Bodies, whose Radius's, or Sides, are A and X, have betwixt themselves a Proportion which is triplicate to the Proportion of A to X, (by *Corollary, Proposition 12.* and by *Proposition 12. and 18. L. 12.*) that is, the same as (per *Defin. 10. l. 5.*) which A hath to B.

And this is that most celebrated Problem which, from *Apollo* and *Delos*, is called the *Deliacal Problem*; because at the Time of a most grievous Pestilence, which was at *Athens*, being consulted, he gave Answer, that the Pestilence would cease, if his Altar, which was of a cubic Form, were doubled. Thus *Valerius Maximus*, *L. 8.*



THEOREMS

SELECTED OUT OF

ARCHIMEDES:

By ANDREW TACQUET,

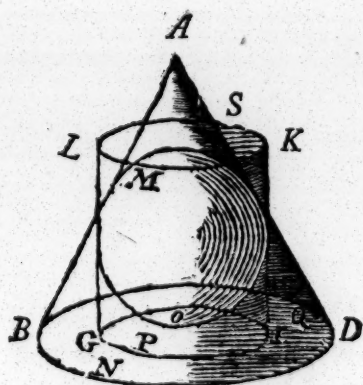
OF THE

SOCIETY of JESUS;

And Demonstrated in a more Easy and Compendious Way.

To which are added,

Some other agreeable Propositions, newly invented, by the same ANDREW TACQUET.



Una tribus Ratio est.

D U B L I N:

Printed in the Year, M DCC LXXI.



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TO THE
R E A D E R.

ALB EIT there have appeared very many most excellent and admirable Men in the Mathematical Sciences; yet the chief Glory of all hath always, by a certain common Consent, been given to Archimedes of Syracuse. Tho' indeed, more there are who commend, than who read him; more who admire, than understand him. The Causes of which Neglect seem to be these, the Bulk and Scarceness of Copies, some Obscurity of the Translation, which is directly made out of the Greek Language, together with the Prolixity and Difficulty of his Demonstrations. I judged therefore that it would be for the Profit of studious Learners, if after my Illustration of the Elements, I should subjoin these Theorems which had been selected by me out of Archimedes, and demonstrated in a much easier and briefer Way. Furthermore, I have selected those, which bring along with them both more of Admiration and of Benefit, and have in my Demonstration took such a Method, that, I hope, he who understands the Elements, will, without
any

To the READER.

any great Labour, comprehend these most excellent Inventions of the Prince of Geometricians. I have also added at the End, thirteen Propositions, and thereby enlarged the Doctrine of Archimedes concerning the Sphere and Cylinder: Where, amongst other Things, I demonstrate, that the Sesquialteral Proportion is continued in the Three Bodies, a Sphere, a Cylinder and Equilateral Cone; both the latter being inscrib'd about the Sphere. Moreover, I have added divers Things here and there, amongst which the 12th Proposition, and the Corollaries of Proposition 14. are the chief; and several Scholiums. Make use of these Discoveries whosoever thou be'st, that art a Candidate of Geometry; and how much thou hast improv'd in Euclid, make Proof of in Archimedes. And when thou perceivest thyself to be fix'd and rais'd upwards in the Contemplation of the most noble Truths, raise up thy Mind, while it is thus already lifted up from these lower Things, yet higher, and direct it to that Truth, which is Original, Eternal, Immense, and is no other than GOD; by the ineffable Vision of whom, I trust we shall hereafter be made eternally Happy. Farewel.

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THEOREMS

Selected out of ARCHIMEDES.

DEFINITIONS.

Or an Explanation of certain Terms.

LET there be a Circle, $BECG$, whose Center is A , *Fig. 23.*
its Diameter BC , which let the right Line EG *Of the Ta-*
cut at right Angles, (but not through the Center) in D . *ble out of*
Let there be drawn from the Center the Radiuses AE , AG . *Archime-*
This being suppos'd, *des.*

NOTE, (1.) That a Sector of a Sphere is that which
is produced from the Sector of the Circle $AECG$, or
 $AEBG$, turn'd round about the Diameter BC .

2. That a Segment, or Portion of a Sphere, is that Part
of it which is produced from the Segment of the Circle
 $ECCG$ or $EBCG$ turn'd round about the same Diameter BC .

3. The Vertex, or Top of the Spherical Portion $EBCG$,
is the Extremity B of the unmov'd Diameter; the Basis,
the Circle describ'd by EG ; the Axis, that Part of the
Diameter BD , which is intercepted betwixt the Top B ,
and D the Center of the Base.

4. When I name the Superficies of a Spherical Portion,
or of a Body inscrib'd in it, or of a Cone, I always under-
stand it without the Base; and when I say the Superficies
of a Cylinder, I mean likewise without the Bases; unless
the Word [*whole*] be adjoin'd to [Superficies;] for then the
Bases also are to be taken in.

Again, when I treat of Cylinders or Cones, I speak of
no other than right ones.

Axioms

Axioms.

Arch.

Fig. 1, 16. 1. **T**HE Circuit of a Polygon inscrib'd in a Circle is less than the Circumference of the Circle. *Prop. 16.*

Fig. 1. 2. The Circuit of the Polygon describ'd about a Circle is greater than the Circumference of the Circle. *Prop. 16.*

Fig. 16. 3. And if a Polygon inscrib'd in a Circle, be turn'd about the Diameter (A E) together with the Circle, the Superficies of the Body produced by the Polygon, will be less than the Superficies of the Sphere. And if a Polygon circumscrib'd about the Circle, be turn'd about the Diameter, together with the Circle, the Superficies of the Body produced by the Polygon, will be greater than the Superficies of the Sphere.

Fig. 17. 4. In like manner the Circuit of a Polygon inscrib'd in a Segment of a Circle (D A F) is less than the Circumference of the Segment. And if a Polygon inscrib'd in the Segment, be together with the Segment (A O) turned round; the Superficies of the Body produced by the Polygon will be less than the Superficies of the Spherical Portion D A F.

Fig. 3, 6. 5. The Superficies of a Prism inscrib'd in a Cylinder, is less than the Superficies of the Cylinder; but the Superficies of the Prism which is circumscrib'd is greater.

Fig. 4, 8. 6. And the Superficies of a Pyramid inscrib'd in a Cone, is less than the Superficies of the Cone; but the Superficies of a circumscrib'd Pyramid is greater.

PROPOSITIONS I, II.

ARE not necessary.

PROP. III. Theorem.

THE Circuits or Polygons circumscrib'd about and inscrib'd in a Circle, do at last end in the Circumference of the Circle. In like manner the Polygons themselves do at last end in the Circle.

Fig. 1. If, to wit, the Arches being bisected without End, more and more Sides be circumscrib'd about and inscrib'd in the Circle.

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Part I. Let there be understood to be inscrib'd in and describ'd about a Circle, regular Polygons; whether it be done to as is set down, *Prop. 12. l. 4.* or as in the present Figure, the Thing will be the same. It is manifest (*per Corol. 1. p. 4. l. 6.*) that FI is to CE (that is, the whole Circuit circumscrib'd, unto the whole Circuit inscrib'd) as IA is to CA . But IC , the Excess of the right Lines IA above CA , becomes at length less than any given Line, if more and more Sides be understood to be infinitely circumscrib'd and inscrib'd; therefore also the Excess of the Circuit circumscrib'd above that which is inscrib'd, will at length become less than any given Line. Therefore much more the Excess of the Circuit circumscrib'd above the Circumference of the Circle, will be less than any given one. In like manner, because I have already shew'd the Defect of the Circuit inscrib'd, whereby it falls short of that which is circumscrib'd, to be less than any given Line: Therefore much more will the Defect of the Circuit inscribed, whereby it falls short of the Circumference of that Circle, become less than any given Line. The Circuits therefore, as well as that which is inscrib'd, as well as that which is circumscrib'd, do at length (*Def. 6. l. 12.*) end in the Circumference. Which was the first Part. To demonstrate these Things further is not worth the while, seeing they are manifest enough.

Part II. Because it hath already been shew'd that the Excess FI , above the Side EC , becomes at length less than any given Line (FI is to EC , as IA to CA ;) therefore also the Excess of the Square of FI , above the Square of EC , will become at length less than any given one. But as the Square of FI , is to the Square of EC , so (*per 20. l. 6.*) is the Polygon circumscrib'd, to that which is inscrib'd. Therefore the Excess of the Polygon circumscrib'd above that which is inscrib'd, will also become at length less than any given one. Therefore much more will the Excess of the Polygon circumscrib'd above the Circle, become at last less than any given one; and consequently, the Defect also of the Polygon inscrib'd, whereby it falls short of the Circle, will at length become less than any given Defect. Therefore Polygons, as well inscrib'd as circumscrib'd, do at last (*Defin. 6. l. 12.*) end in the Circle. Which was the second Part.

P R O P.

PROP. IV. Theorem.

(c) Per
Defin. 3.
l. 4.
Fig. 1.

A Regular (c) Polygon (FINT R) circumscrib'd about a Circle, is equal to a Triangle whose Base is the Circuit of the Polygon, and its Height the Radius of the Circle.

And a regular Polygon inscrib'd in a Circle is equal to a Triangle, which hath for its Base the Circuit of the Polygon, and for its Height the Perpendicular (A O) let down into one Side from the Center.

Part I. The Radius A B, drawn to the Point of Contact, is (per 18. l. 3.) perpendicular to the Tangent I F. Wherefore, if the right Lines A F, A I, A N, &c. being drawn, the Polygon be resolv'd into Triangles; the Radius A B will be the common Altitude of all; and consequently it is manifest that the Triangles are equal. Therefore a Triangle, which hath its Base equal to the Sum of the Sides F I, I N, N T, &c. and A B for its Altitude, will (as is manifest from 1. l. 6.) be equal to them all, that is, to the whole Polygon circumscrib'd.

Part II. This may be concluded by the same reasoning as the other.

Arch

PROP. V. Theorem.

Fig. 2.

A Circle is equal to a Triangle, which hath for its Base the Circumference, and for its Height the Semi-diameter of the Circle.

Regular Polygons circumscrib'd about a Circle, and Triangles which have for their Bases the Circuit of the Polygon, and for their Altitude the Radius of the Circle, are always (by the foregoing Proposition) equal. But Polygons circumscrib'd infinitely about the Circle end in the Circle, (by the third of this Book;) and in like manner Triangles (as I will shew by and by) which have for their Base the Circuit of the circumscrib'd Polygon, and for their Altitude the Radius A B, at last end in a Triangle, which hath the Circum-

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Circumference for its Base, and for its Altitude the Radius AB. Therefore (by the first) a Circle and a Triangle, which hath the Circumference for its Base, and the Radius for its Altitude are equal.

But that Triangles contain'd under the Circuit of the Polygon, and the Radius of the Circle, end at last in a Triangle, which is contain'd under the Circumference and the Radius, I thus shew. Triangles under the Circuit of the Circumscrib'd Polygon, and the Radius AB, are to the Triangle, which is under the Circumference and the Radius AB, (by 1. 1. 6) as Base to Base, that is, as the Circuit of the Polygon to the Circumference; since this Triangle and the other have a common Altitude. But the Circuit of the Polygon (by the third) ends in the Circumference. Therefore the other Triangles end in this.

Corollaries.

FROM this and 41. 1. 1. it is manifest that a Rectangle under the Radius, and half the Circumference, is equal to the Circle; that one under the Radius, and the whole Circumference, is double; that one under the whole Circumference and the whole Diameter is quadruple thereto.

1. A Circle is to an inscrib'd Square, as half the Circumference (CDE) is to the Diameter; but to a Square circumscrib'd, as the fourth Part of the Circumference is to the Diameter. *Fig. 5. 1. 4.*

For the Rectangle under CDE, and the Radius CA or AF, that is, (by the foregoing *Corollary*) the whole Circle to the Rectangle GFCE, to wit, the Rectangle under FG and CF (that is, to the inscrib'd Square BCDE) as (by 1. 1. 6.) CDE, half the Circumference is to FG or CE the Diameter; which was the first Thing. And consequently the Circle is to the double of the Rectangle GFCE, that is, to FH, the Circumscrib'd Square) as CDE is to the double of the Diameter CE, or as the Quadrant CD to the Diameter CE.

[3. "Of Figures which are of equal Circumferences, *Fig. 30.* the Circle is the most capacious. Let the Circumference of any Polygon whatsoever (as for Instance of a Square) EFGHI be equal to the Circumference of the Circle. I say, that the Area of the Circle is greater than that of the Polygon. For the Area of the Circle is equal to a
O " Triangle,

" Triangle, whose Base is the Circumference, and its
 " Altitude the Semi-diameter F A: And the Area of the P
 " lygon is equal to a Triangle, whose Base is the Comp
 " of the Polygon; which, by the Hypothesis, is equal
 " the Circumference of the Circle, and which hath for
 " Altitude the Perpendicular F O, let down from the C
 " ter of the Circle unto the Side of the Polygon, whi
 " since it is always less than the Radius of the Circle, i
 " manifest that the Area of the Polygon is less than
 " Area of the Circle. Q. E. D. And in like mann
 " amongst all solid Figures contain'd under equal Surfa
 " the Sphere may be demonstrat'd to be the most ca
 " cious.]

Arch. Prop.

PROP. VI. Theorem.

THE Circumference of a Circle contains
 Diameter less than thrice and one seven
 (or $\frac{10}{7}$;) and more than thrice and $\frac{1}{7}$.

For the Demonstration of this Theorem, *Archimedes*
 sumes regular Polygons, one circumscribed about a Circle
 the other inscribed, and both of them of 96 Sides. And
 then he shews that the 96 inscribed about a Circle, do
 tain the Diameter less than thrice and one seventh, and
 sequently that the Circumference which is less than the
 doth also contain the Diameter less than thrice and one
 venth. But the 96 Sides inscribed in the Circumference
 (and consequently the Circumference also which is greater
 than them) doth contain the Diameter more than three
 times $\frac{1}{7}$. But this Demonstration is too long to be brought
 in this Place. Nay, if we minded to extend our Geometrical
 Reasoning to Polygons of more Sides still, we may con
 tract the Limits even now set more and more without
 mit, and so come nearer and nearer for ever to the true
 portion. This hath been perform'd by *Ludolph a German*
Grimberger, Metius, Snellius and others. The chief
 portions hitherto found I shall here subjoin.

" [However, since a Tangent of 30 Degrees, multipli
 " by 12, gives the Circuit of a circumscribed Hexagon
 " and a Sine of 30 Degrees multiplied by 12, gives
 " Circuit of an Hexagon, which is inscribed: Forasmuch
 " also as in like manner the Tangent of half a Degree

multiplied by 720, yields the Circuit of a circumscribed Polygon of 360 Sides; and the Sine of half a Degree, the Circuit of an inscribed Polygon of 360 Sides; and so on for ever: It will not be difficult to understand, by what Means many such Numbers may be found, out of the now given Tables of Sines and Tangents.]

The first Proportion, which is that of *Archimedes*, is thus:

The Diameter 7.

The Circumf. is 22; which is greater than the true.

The Diameter 71.

The Circumf. is 223; less than the true one.

The Proportion of 22 to 7, and 223 to 71, if they be reduced to a common Consequent, (which is done after the same manner, in which Fractions are reduced to the same denomination) will be thus, 1562 to 497, and 1561 to 497.

Therefore the Diameter being suppos'd 497 Parts, the Circumference greater than the true one, will be 1562; and the Circumference less than the true, 1561.

Both of them therefore differ from the true, by a Quantity less than $\frac{1}{497}$ Part of the Diameter. But if the Proportion of 7 to 22, and 71 to 223, be reduced to a common consequent, there will arise the Proportions of 1561 to 496, and of 1562 to 4906.

Therefore the Circumference being suppos'd to be 4906 Parts, the Diameter less than the true, will be 1561, the Diameter greater than the true, 1562.

Both therefore differ from the true Diameter by a Quantity less than $\frac{1}{4906}$ Part of the Circumference.

The Proportion delivered by *Metius* is much more accurate than this of *Archimedes*. According to this,

The Diameter is 113.

The Circumference 355.

Amongst all Proportions consisting of small Numbers, none comes nearer to the true one; for from this, the Diameter being suppos'd of 10,000,000 Parts, the Circumference comes to be of 31,415,929, which differs from the true one only in the first Figure 9, and this by an Excess, but a little, greater than two ten-millioneth Parts of the Diameter.

But more exact than both, is that double Proportion of *Ludolphus à Ceulen*; the former of which consists of 2 Figures, and the latter of 36.

The Diameter.

100,000,000,000,000,000,000.

The Circumf. greater than the true.

314,159265,358979,323847.

The Circumf. less than the true.

314,159265,358979,323846.

The Difference of both the Circumferences is one Part of the Diameter denominated from a Number which consists of a Unity and 20 Cyphers; and consequently, well this as that, differs from the true Circumference by a Quantity less than is the said small Part of the Diameter to wit, one hundredth of a millioneth of a millioneth of a millioneth Part.

The Diameter.

100600,000000,000000,000000,000000,000000.

The Circumf. greater than the true,

314159,265358,979323,846264,338327,950289.

The Circumf. less than the true,

314159,265358,979323,846264,338327,950288.

The Difference of both the Circumferences, between which is the true one, is that small Part of the Diameter denominated from a Number which consists of Unity and 35 Cyphers; which small Part bears a less Proportion to the whole Diameter, than one little Grain of Sand doth to the whole Globe of the Earth. For the whole Globe of the Earth doth not consist of so many little Grains of Sand as are the little Parts of the said Sort which are contained in the Diameter.

It is needless to go any further. Nevertheless you may proceed infinitely, if you be minded to continue Geometrical Reasoning, an expedite Method of which is delivered *Snellius*.

[The Circumference being suppos'd of

1,000000,000000,000000,000000,000000,000000 Part

The Diameter will be as near as may be, of

0,318309,886183,790671,537767,925745,028724 Part

Scholium

Scholium.

THE most excellent Advantages of the Proportion now delivered, are these which follow.

The Invention of the Diameter from the Circumference.

LET the greater Term of one of the Proportions which have been now delivered in the first Place, the lesser in the second, the Circumference in the third; by these three Numbers let there be sought by the Golden Rule a fourth proportional. That is the Diameter sought.

As if the Circumference of the greatest Circle of the Earth be suppos'd to contain 25000 *English* Miles of 5280 Feet each, and the Diameter be sought; the Terms will stand thus,

$$355—113—25000—7854.$$

Multiply now the second by the third, and divide the Product by the first; and there will arise 7854 Miles for the Diameter of the Globe of the Earth.

The finding out of the Circumference from the Diameter.

LET the lesser Term of one of the Proportions above delivered be set in the first Place; the greater in the second; the known Diameter in the third: And by these three Numbers, let there be sought a fourth Proportional, that will give the sought Circumference.

As if the Diameter of the Globe of the Earth be suppos'd to contain 7854 *English* Miles; and the Circuit is sought; the Terms will stand thus,

$$113—355—7854—25000.$$

Then multiply the second by the third, and divide the Product by the first; there will arise 25000 Miles for the Circumference of the Globe of the Earth.

How little this Circumference exceeds the true one, was said above; to wit, by an Excess but a little greater than two ten millioneth Particles of the Earth's Diameter; that is, by 9 or 10 Feet. But if we use the *Ludolpbian* Proportion, even the former, the Terms whereof consist

of 21 Figures; there will be found a Circumference insensibly differing from the true, not only when the given Diameter is of 7854 Miles, such as is the Diameter of the Earth; but also although the Diameter be suppos'd, of a 100 Millions of those Miles. For this being suppos'd, there will arise a Circumference differing from the true one by a Quantity about one hundred millioneth Part of a Foot. But if to find out the Circumference of the Globe of the Earth we make use of the Proportion of *Archimedes*, the Interval of the two Circumferences, the one greater, the other less than the true one, will exceed 20 Miles. *Archimedes* his Proportion therefore is not to be used but in small Measures; nay, it will always be expedient to use that of *Metius* which both consists of small Terms, and is above a 100 Times more exact.

The measuring of a Circle.

THE Semi-diameter multiplied by half the Circumference, produceth the Area of the Circle; as is manifest from *Corollary 1. Proposition 5.* of this Book.

As if the Semi-diameter of the Earth, which containeth 3927 Miles, be multiplied by half its Circumference, to wit, by 12500, there will arise 49,075500, Miles Squares for the Area of the greatest Circle of the Earth. The Difference of the circular Area thus found from the true Area had, if the Difference of half this found Circumference from the true half Circumference be multiplied by the given Semi-diameter; or the Difference of this Semi-diameter from the true, be multiplied by the given Semi-circumference.

The Mensuration of Cylinders and Cones.

Put this here, because it depends upon the Mensuration of a Circle. A Cylinder therefore, and any Prism whatsoever, is produced from the Altitude multiplied by the Base. A Cone and Pyramid, from the third Part of the Altitude multiplied by the Base; for they are third Parts of Cylinders and Prisms, having the same Base and Altitude with them by 10, and 7. *l.* 12.

Let the Base of a Cylinder or Cone, be of 50 Squares Feet, and the Height of 100 Feet. Multiply 100 by 50 and there arise 5000 cubic Feet for the Solidity of the Cylinder.

der. Multiply the third Part of the Altitude 100, which is $33\frac{1}{3}$ by 50, there arise $1666\frac{2}{3}$ cubical Feet for the Solidity of the Cone.

PROP. VII. Theorem.

THE Circumferences of Circles have the same Proportion betwixt themselves which their Diameters have. *Fig. 6, 7. 12.*

For the Circuits of like Polygons, which may be inscribed in a Circle without End, are always betwixt themselves, as the Diameters A F and I C (by Corollary, p. 1. l. 1.) But these Circuits (by the 3d Proposition of this Book) are at length in the Circumference. Therefore their Circumferences also are betwixt themselves as their Diameters. *E. D.*

PROP. VIII. Theorem.

THE Superficies of a Prism, as well that which is circumscrib'd about, as that which is inscrib'd in a Cylinder, is equal to a Rectangle, whose Height is the Side of the Cylinder, but its Base equal to the Circuit of the Base of the Prism.

Part I. The Superficies of the circumscribed Prism *Fig. 3.* touches the Cylinder according to the Lines E A, N F, &c. which are the Sides of the Cylinder; but these (because by the Hypothesis the Cylinder is a right one) are right to the Plane of the Base, and consequently right also (by Definition 3. L. 11.) to the Lines C G, G M, &c. But they are also equal betwixt themselves. Therefore one Side of the Cylinder is the common Height of all the Rectangles E O, O M, M H, &c. Therefore the Superficies of the circumscribed Prism is equal (as is manifest from 1. L. 6.) to a Rectangle contain'd under the Circuit of the Base of the Prism, and the Side of the Prism or Cylinder.

Part II. The Reason of this is the same. For the Side of the Cylinder is again the common Altitude of the Rectangles B D I K, K I P Q, &c. which constitute the Superficies of the inscribed Prism.

PROP.

Arch Prop IX PROP. IX. Theorem.

Fig. 4. **T**HE Superficies of a regular Pyramid circumscrib'd about a right Cone, is equal to a Triangle, which hath for its Base the Circumference (FHL D) of the pyramidal Base, but its Height the Side of the Cone (BG.)

And the Superficies of a regular Pyramid inscribed in a right Cone, is equal to a Triangle which hath for its Base the Circumference of the pyramidal Base, but for its Height the Perpendicular (BO) let down from the Top unto a Side of the Base.

Part I. Let there be drawn unto the Contacts G, M, the right Lines BG, BK, BM. These will all be Sides of a right Cone, and consequently equal. And, because, (by the Hypothesis) the Axis BA is perpendicular to the Plane of the Base FKD, the Plane also GBA (per 1. 11.) will be perpendicular to the Plane FKD. BGHG (per 18. 1. 3.) is perpendicular to AG, the common Section of the Planes FKD and GBA. Therefore HG (as is gathered from Defn. 4. 1. 11.) is also perpendicular to the Plane GBA. And consequently is also perpendicular to BG. Therefore the Side GB of the Cone, is the Height of the Triangle FBH. In the same manner the Side of the Cone will be the Height of the rest, HBLD, &c. Therefore the Triangle comprehended under the Circumference FHL D, and the Side of the Cone, is equal to the Superficies of a Pyramid circumscribed, without the Base. Which was the first Part.

II. The Demonstration of this Part is almost the same with that of the former.

PROP. X. Theorem.

THE Superficies of a regular Prism circumscribed about a right Cylinder, ends (Def. 6. 1. 12) in the Superficies of the Cylinder; and the Superficies

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of a Pyramid circumscribed about a right Cone ends in the Superficies of the Cone.

Part I. The Superficies of regular Prisms described Fig. 3. about and inscribed in a Cylinder without end will have at last a Difference betwixt themselves less than any which can be given. Much more therefore will the Superficies of a circumscribed Prism differ from the Superficies of the Cylinder, which is in the middle between the inscribed and circumscribed Superficies, by a Difference less than any given one whatsoever; that is. (*Def. 6. l. 12.*) will end in the cylindrical Superficies, whilst it continually exceeds it less and less.

Part II. This may be shewed in the same manner from Fig. 4. the 9th and 3d of this.

In the Figures there are only exhibited the Halves of the Cylinder and Cone, lest a Multitude of Lines should breed Confusion. But the Cylinder and Cone are to be conceiv'd in the Mind entire, and as having these circumscribed Prisms and Pyramids encompassing them. For thus it more clearly appears that plain Surfaces circumscribed are greater, according to the third Axiom.

A Lemma to the following Proposition.

LET AB, CD, EF , be proportional, and let KB be Fig. 7. half AB , and EG , double EF ; KB, CD, EG , will also be proportional.

The right Line KB is to AB , as EF is to EG . Therefore the Rectangle KB, EG (*per 16. l. 6.*) is equal to the Rectangle AB, EF . But this (*by 17. l. 6.*) is equal to the Square of CD . Therefore also the Rectangle KB, EG is equal to the Square of CD . Therefore (*by 17. l. 6.*) KB, CD, EG are proportional.

PROP. XI. Theorem.

A Circle, whose Radius (GH) is a mean Proportional betwixt the Side of a right Cylinder (BC) and the Diameter of the Base (BD) is equal to the cylindrical Superficies. *Arch Prop XIV* Fig. 5, 6.

Let the regular, and consequently like Polygons, NM, RS , be understood to be circumscribed about the Circles ABN, GPH ; and upon the Polygon NM , let a Prism be

be conceiv'd to be erected, with which the Cylinder is circumscribed. Because BD , GH , BC are, by the Hypothesis, proportional, AD also, (or AN) GH , and the double of BC will, by the *Lemma*, be proportional. Now the Triangle contain'd under AN , and the Circuit of the Polygon MN is equal to the Polygon circumscribed NM (by the fourth of this Book.) And the Rectangle under BC , or EF , and the same Circuit NM , (that is, as is manifest from 41. *L. 1.* the Triangle under the Circuit NM , and the double of BC) is equal (by the eighth of this Book) to the Superficies of a Prism circumscrib'd about the Cylinder. But a Triangle under the Circuit NM and AN , is to the Triangle under the Circuit NM , and the double of BC (by 1. *L. 6.*) as AN is to the double of BC . Therefore the Polygon NM also is to the Superficies of a Prism circumscrib'd about a Cylinder, as AN is to the double of BC . But because I have already shewed AN , GH , and the double of BC to be proportional, the Proportion of AN to the double of BC is (by *Def. 10. l. 5.*) duplicate to the Proportion of AN to GH . Therefore the Polygon NM hath to the Superficies of the Prism a Proportion duplicate to the Proportion of AN to GH . But the Polygon NM hath also to the Polygon like to it, $GRQS$, a Proportion duplicate to the Proportion of AN to GH , as is easily gathered out of 1. *L. 12.* Therefore the Polygon NM hath the same Proportion to the Superficies of the Prism, which it hath to the Polygon $GRQS$; which consequently is equal to the Superficies of the Prism. In the same manner I might shew, that the prismatic Superficies, which are circumscribable infinitely about the Cylinder, are always equal to the Polygons which may be circumscribed infinitely about the Circle GPH . Wherefore seeing both the prismatic Superficies (by the 10th of this) end in the Surface of the Cylinder, and the Polygons in the Circle GPH (by the 3d of this) the Superficies of the Cylinder also will be equal to the Circle GPH . *Q. E. D.*

From this admirable Theorem, a Circle is presented, which is equal to a cylindrical Superficies.

Corollaries

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Corollaries.

THE Superficies of a right Cylinder is equal to a Fig. 5, 6:
*Rectangle contained under the Side (BC) and the
 Circumference of the Base.*

The double of BC (as hath been shew'd above) is to GH, as GH is to BA, or AN; that is, (by the 7th of this) as the Circumference P is to the Circumference BN. Therefore the Triangle under the first, to wit, the double of BC; and the fourth, to wit, the Circumference BN, is equal to a Triangle under the second GH, and the third, to wit, the Circumference P, (as appears from 16. l. 6.) But the Triangle under GH, and the Circumference P, is (by the 5th of this) equal to the Circle GPH, that is, (by the 11th of this) to the cylindrical Superficies. Therefore also the Triangle under the double of BC, and the Circumference BN, (that is, as appears from 41. l. 1. the Rectangle which is under BC and the Circumference BN) will be equal to the cylindrical Superficies. Q. E. D.

From this Corollary it is manifest, that the Properties of Rectangles are common to them with cylindrical Superficies. Therefore let this be Corollary 2.

2. The cylindrical Superficies (BM, QN) which are of Fig. 20.
 the same Height, are betwixt themselves as the Diameters l. 12.
 of their Bases (BF, QR.)

For the Rectangles under the Circumference (CL, SE) and the equal right Lines FM, RN, to which (by Coroll. 1.) the cylindrical Superficies are equal, are betwixt themselves (by 1. l. 6.) as the Bases, to wit, the Circumferences CL, SE; that is, as the Diameters BF, QR, (by the 11th of this)

3. The cylindrical Superficies (CI, AR) which have equal Bases, are betwixt themselves, as their Altitudes (TI, BR)

For the Rectangles contain'd under the equal Circumferences GB, MQ, and the Sides TI, BR, to which (by l. 12. Coroll. 1.) the cylindrical Surfaces are equal, are betwixt themselves (by 1. l. 6.) as TI, BR.

4. Like cylindrical Surfaces (BM, RI) have betwixt Fig. 20, 21.
 themselves a Proportion duplicate to that which (BF, QR) l. 12.
 the Diameters of the Bases have.

Seeing

Seeing the Cylinders are suppos'd to be like, MF will be to IQ (by *Defn.* 4. *l.* 12.) as BF is to QR ; that is (by the 7th of this) as the Circumference CL to the Circumference SE . Wherefore the Rectangles also which are contain'd under the Circumferences CL , SE , and the Sides MF , IQ , will be like; and consequently they will have betwixt themselves (by 20. *l.* 6.) a Proportion duplicate to that which MF hath to IQ ; that is, BF to QR . Therefore the cylindrical Surfaces also have the same.

*The same
Figure.*

*Fig. 24, 25.
l. 12.*

5. Cylindrical Surfaces (BM , RI) have betwixt themselves a Proportion compounded of the Proportions of the Sides (FM , IQ) and the Diameters of the Bases (BF , QR), as is manifest from 23. *l.* 6. and the 7th of this.

6. If cylindrical Surfaces (AR , FD) be equal, as the Diameter (AB) is to the Diameter (FN) so reciprocally (by 14. *l.* 6.) the Altitude (FH) will be to the Altitude (BR) and conversly.

7. Lastly, from the same first *Corollary* is had the Measure of a cylindrical Superficies; to wit, if the Circumference of the Base be multiplied by the Altitude. As if the Altitude be of 20 Feet, the Circumference of the Base of 6; multiply 20 by 6, there arise 120 Square Feet for the cylindrical Superficies.

PROP. XII. Theorem.

THE Superficies of a right Cylinder is to the Base (ABN) as the Side of the Cylinder (CB) is to (BO) the fourth part of the Diameter of the Base.

Let GH be a mean Proportional betwixt BC the Height, and BD the Diameter of the Base, and consequently (by *Lemma* before *Prop.* 11. of this) a mean Proportional betwixt AN and the double of BC . The Circle GPH , of the Radius GH , is (by the 11th of this) equal to the Curve cylindrical Superficies CD . But the Circle GPH hath to the Base of the Cylinder ABN a Proportion duplicate (by 2. *l.* 12) to the Proportion of GH to AN ; that is, the same which the double of BC hath to the Radius BA (by the Hypothesis, and *Def.* 10. *l.* 5.) that is, the same which BC hath to BO , the fourth Part of the Diameter.

meter. Therefore the cylindrical Superficies also is to the Base ABN , as BC is to BO , the fourth Part of the Diameter. *Q. E. D.*

Corollary:

THE Superficies of a Cylinder which hath its Sides equal to the Diameter of its Base, is four fold of the Base. But if the Side be a fourth Part of the Diameter of the Base, the Superficies of the Cylinder will be equal to the Base. Both these are manifest from the Proposition.

Arch. Prop. XV B. I.

PROP. XIII. Theorem.

A Circle, whose Radius (OL) is a mean Proportional betwixt the Side (BC) of a right Cone, and the Radius of the Base (AC) is equal to the Lateral Superficies.

Let regular Polygons, EF , IN , be understood to be circumscrib'd about the Circles ACG , OPL , and a Pyramid circumscrib'd about the Cone to be erected upon the Polygon EF .

Because, by the Hypothesis, AC , or AG , is to OL ; as OL is to BC , the Proportion of AG to BC , will (Defin. 10. l. 5.) be duplicate to the Proportion of AG to OL . But as AG is to BC , so is the Triangle under AG , and the Circuit EF , to the Triangle under BC and the same Circuit EF . Therefore the Proportion of the Triangle under AG , and the Circuit EF , to the Triangle under BC , and the same Circuit, is also duplicate to the proportion of AG to OL . But the Triangle under AG and the Circuit EF is equal to the Polygon LF (by the 9th of this :) And the Triangle under BC and the same Circuit EF (by the 9th of this) is equal to the Superficies of the circumscribed Pyramid. Therefore the Proportion of the Polygon EF , to the Superficies of the Pyramid, is also duplicate to the Proportion of AG to OL . But the proportion of the Polygon EF to the Polygon IN , which, by the Construction, like to it, is (per 1. l. 12.) also duplicate to the Proportion of AG to OL . Therefore the Polygon

Polygon E F hath the same Proportion to the Superficies of the Pyramid, and to the Polygon I N, which consequently are equal. In the same manner I might shew, that the Superficies of Pyramids, which may be circumscribed about a Cone infinitely more and more, are always equal to Polygons which may be circumscribed infinitely about the Circle O P L. Wherefore seeing both the Surfaces of Pyramids (by the 10th of this) do at last end in the Surface of the Cone and Polygons (by the 3d of this) in the Circle O P L, the Superficies of the Cone and the Circle O P L, shall likewise be equal. *Q. E. D.*

From this excellent Theorem a Circle is found which is equal to a conical Surface.

Corollaries.

Fig. 9, 8. 1. **T**HE Superficies of a right Cone, is equal to a Triangle comprehended under the Side of the Cone [B C] and the Circumference of the Base [C G.]

Let O L, the Radius, be a mean Proportional betwixt A C and B C. Then because (by the 7th of this) the Circumference C G is to the Circumference P, as the Radius A G is to the Radius O L; that is, by the Hypothesis, as O L is to B C; the Triangle under the first, to wit, the Triangle under the Circumference C G, and under the fourth B C (as appears from 16. l. 6.) will be equal to the Triangle under the second, to wit, the Triangle under the Circumference P, and the third O L; that is, (by the 5th of this) to the Circle O P L; that is, to the conical Superficies (by the 13th of this) B C D. *Q. E. D.*

From this Corollary it appears that conical Surfaces have the same Properties with Triangles. And so it follows.

Fig. 20, 21. l. 12. 2. That the conical Superficies [B A F, Q X R] having their Sides [B A, Q X] equal, are betwixt themselves as the Diameters of their Bases [B F, Q R.] And,

Fig. 23, 24. l. 12. 3. Those which have equal Bases, C F T, A Z B, are betwixt themselves as their Sides [C F, A Z.] And,

Fig. 20, 21. l. 12. 4. Those conical Superficies [B A F, Q Z R] which are like, have betwixt themselves a Proportion duplicate to that which is betwixt the Diameters of the Bases. And,

The same Figure. 5. All conical Superficies whatsoever have betwixt themselves a Proportion which is compounded of the Proportion

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of the Sides [B A, Q Z] and of the Diameters [B F, Q R] which are in the Bases. And,

6. Those which are equal have their Sides and the Diameters of the Bases reciprocally proportional; and those which have them so, are equal.

All which is demonstrated from *Corollary 1.* as above we deduced the *Corollaries* concerning the cylindrical Surface out of the first *Corollary* there.

7. Lastly, we may measure a right conical Surface, if *Fig. 25.* we multiply the Side F C by half the Circumference of the Base. As if the Side be of 5 Feet, the Circumference of the Base of 20; multiply 5 by 10, and there will arise 50 Square Feet for the conical Superficies. The Demonstration is manifest from the same first *Corollary*.

PROP. XIV. Theorem.

THE Superficies of a right Cone is to the Base, *Fig. 8, 9.* as the Side (B C) is to (A C) the Radius of the Base.

Between the Side B C and A C, the Radius of the Base, let O L be a mean Proportional. Therefore the Proportion of B C to A C, is duplicate to the Proportion of O L to A C (*Defn. 10. l. 5.*) Now (by the 13th of this) a Circle of the Radius O L is equal to the conical Superficies C B D. But the Proportion of this to A C G, the Base of the Cone is (by 2. l. 12.) duplicate to the Proportion of O L to A C; and consequently is the same with the Proportion of B C to A C. Therefore the Proportion of the conical Superficies C B D is to the Base A C G, as B C is to A C. Q. E. D.

Corollaries.

THE Superficies of a right Cone produced by an *Fig. 27.* equilateral Triangle turned about the Perpendicular (K A) is double to the Base (Q T.)

For the Side K B is equal to B D, and consequently double to the half of it A B, which is the Radius of the Base.

2. The

Fig. 24.

2. *The Superficies of a Cone produced by a right angled equicrural Triangle ($E B D$) is to the Base as in a Square the Diameter is to the Side.*

For the Perpendicular $B A$ being drawn, the right Angle B (by 26. l. 1.) is bisected, and consequently $A B D$ is half right. But $A D B$ is also an half right Angle (by Coroll. 11. p. 32. l. 1.) Therefore $D A$, $B A$, are (by 6. l. 1.) equal; and consequently $B D$ is the Diameter of the Square $A K$, whereof $A D$ is the Side. Now the same $A D$ is the Semi-diameter of the Base $P T$, seeing the Perpendicular $A B$ (by 26. l. 1.) bisects $E D$. From which, and the Fourteenth, the Corollary is manifest.

Fig. 24.

3. *The Superficies of the right Cylinder ($G K$) is to the Superficies of the right Cone ($G B N$) as the Side of the Cylinder is to half the Side of the Cone.*

For the Superficies of the Cone, $G B N$, is to the Base $M I$, as the Side $B N$ is to $Q N$, the Semi-diameter of the Base (by the 14th of this) that is, as half the Side $B N$ is to the fourth Part of the Diameter $G N$. But the Base $M I$ (by the 12th of this) is to the Superficies of the Cylinder $G K$, as the fourth Part of the Diameter is to $N K$ the Side of the Cylinder. By Equality of Proportion therefore the conical Superficies $G B N$ is to the cylindrical Superficies $G K$, as half the Side of the Cone is to $N K$ the Side of the Cylinder. *Q. E. D.*

A Lemma to what follows.

Fig. 10.

IN a Triangle, as $N P V$, let there be drawn $Q D$ parallel to $N V$.

I say, that the Rectangle under $P N$ and $N V$ is equal to the Rectangle under $P Q$, $Q D$, together with the Rectangle under $N Q$, and the two $N V$, $Q D$ put together.

Draw $N A$ perpendicular to the Side $N P$, and equal to $N V$; and the Rectangle $N O$ being completed, let the Diameter $P A$ be drawn. Then from Q let there be drawn

Q E

E parallel to NA, which may cut PA in B. Through
 let CF be drawn parallel to NP. Because AN is equal
 NV, it is manifest that QB also is equal to QD, (from
 Corollary 1. p. 4. l. 6.) Therefore the Rectangle ON is
 the Rectangle PN V, and FQ is PQD. It remains, that
 we prove that the Rectangles OB, EC, BN, are equal to
 the Rectangle under NQ, and the two NA, BQ; that
 to the Rectangle under NQ, and the two Lines NV,
 D. But that is manifest; for the Rectangle under NQ
 and NA, QB is equal (per 1. l. 2.) to their three Rectan-
 gles, that under NQ and CA (that is, the Space EC)
 and that under NQ and NC (that is, the Space BN) and
 that under NQ and QB, that is again the Space BN,
 and consequently the Space OB, which (per 43. l. 1.) is
 equal to BN. The Proposition therefore is manifest.

PROP. XV. Theorem.

Arch. Prop. XVII

If a right Cone be cut by the Plane QSB, paral-
 lel to NZO; I say, that the Circle GHM
 whose Radius GH is a Mean betwixt Part of the
 NZ, and QD, NV taken together, the Ra-
 dius's of the Circles QSB, NZO, is equal to the
 conical Surface intercepted betwixt the parallel Circles
 QSB, NZO.

Let GF be the Mean betwixt PN and NV. Likewise
 GK be the Mean betwixt PQ and QD; and let there
 be described the Circles GFL, GKT. This (by the 13th
 of this) will be equal to the conic Superficies QPB, and
 together to the Superficies NOP. The Rectangle PN V
 (by the Lemma) is equal to the Rectangle PQD, together
 with the Rectangle under NQ and NVQD, taken toge-
 ther. But because (by the Construction) GF is a mean
 proportional betwixt PN, NV; the Rectangle PN V is
 equal to the Square of GF (by 17. l. 6.) And because
 GK is (by the Construction) a Mean betwixt PQ, QD,
 the Rectangle (by 17. l. 6.) PQD is equal to the Square
 of GK: And because GH, by the Hypothesis, is a Mean
 betwixt QN and QD, NV taken together, the Rectangle
 (by 17. l. 6.) under QN, and QD, NV, taken together,

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is

is equal to the Square of GH . Therefore the Square of GF is also equal to the Square of GH , and to that of GK . Therefore seeing Circles are betwixt themselves (b. 2. l. 12.) as the Square of the Radius's, the Circle GLF will also be equal to the two Circles GKT , GHM taken together. But (by the 13th of this) the Circle GLF is equal to the conical Superficies NPO . Therefore the conical Superficies NPO is also equal to the two Circles GKT and GHM . But QPR , one Part of the Superficies NPO , is (by the same) equal to the Circle GKT . Therefore the remaining Part, which is comprehended betwixt the parallel Circles ZZ , SS , is equal to the Circle GHM . $Q.E.D.$

A Lemma to what follows.

Fig. 13. **R**IGHT Lines (BH , CG) which in the Circle intercept equal Arches (BC , HG) are parallel.

For let CH be drawn. Because the Arches BC , HG are, by the Hypothesis, equal; the alternate Angles also (29. l. 3.) BHC , GCH , will be equal. Therefore (28. l. 1.) BH , and CG are parallel. $Q.E.D.$

PROP. XVI. Theorem.

Fig. 13.

LET there be inscribed in a Circle a regular Figure of an even Number of Sides, and let it be equilateral; let EB be drawn from the Extremity of the Diameter unto B , the End of two Sides next to the Diameter: and let the right Lines BH , CG , DF , join the Angles which are equally distant from A .

I say, that the Rectangle contained under the Diameter AE , and the Subtense EB , is equal to the Rectangle of one Side of the inscribed Figure AB , BC , &c. and of all the joining Lines BH , CG , DF , taken together.

Draw CH , DG : Because BH , CG , DF intercept (26. l. 3.) equal Arches BC , HG ; CD , GF , &c.

lines (by the *Lemma*) will be parallel. By the same Argument BA, CH, DG, EF , are parallel. All the Triangles therefore (by 27, and 15. *l.* 1.) $BAK, KHL, LCM, MGN, NDO, OFE$, are Equi-angular. Therefore (by 1. 6.) as BK is to KA , so is HK to KL ; and as HK to KL , so is CM to ML ; and as CM is to ML , so is DM to MN ; and as GM is to MN , so is DO to ON ; and as DO is to ON , so is FO to OE . Therefore (by 1. 5.) as one of the Antecedents BK , is to one of the Consequents KA ; so all the Antecedents BK, KH, CM, MG, DO, OF , (that is, all the joining Lines BH, CG, DF) are to all the Consequents AK, KL, LM, MN, NO, OE , (that is, to the Diameter AE .) But (by 8. 6.) as BK is to AK , so is EB to BA . Therefore as all the joining Lines BH, CG, DF , are to AE , so is EB to BA . Therefore (by 16. *l.* 6.) the Rectangle under BA on one side, and all the joining Lines BH, CG, DF , on the other, is equal to the Rectangle which is under AE and EB , *E. D.*

PROP. XVII. Theorem.

Arch. Prop. XXIII

Let there be inscribed in DAF a Segment of Fig. 14. a Circle, whose Base DF is perpendicular to the Diameter AOE , a Figure equilateral, and an even Number of Sides; and let there be drawn, as in the foregoing, the Line EB . I say, that the Rectangle comprehended under EB and AO , part of the Diameter, which is the Axis of the Segment; is equal to the Rectangle, which is under one Side of the inscribed Figure, and all the joining Lines $BH, CG, \&c.$ taken together with half the Base DF .

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The Demonstration is the same with that of the foregoing.

Lemma I. to what follows.

Let there be inscribed in the greatest Circle of a Sphere Fig. 13. a regular Figure, which hath its Sides measured by the Number Four; and stands about the Axis AE ; which is remaining unmov'd, let the Circle be turn'd round together with the Figure:

P 2

I say,

I say, that there will be inscribed in the Sphere a Bo
contain'd under conical Superficies.

It is manifest (see *Defn. 2. l 12.*) that B A, H A, likewise D E, E F, describe entire Superficies of right Cones. Then, because the Lines C B, G H, and G F, C D, being produced, do concur on both Sides in the same Point of Diameter A E, which is in like manner to be drawn on the other Side, and cuts the joining Lines perpendicularly: It is also manifest that the said Lines C B, G H, &c. do describe Parts of right conical Surfaces which are intercepted betwixt the parallel Circles, which the Tops of the Angles B, C, D, describe in the Spherical Superficies.

Lemma 2.

LET DAF be the greatest Section of a Segment of a Sphere whose Axis is A O. Let there be inscribed in this a Figure having all the Sides equal, the Base excepted, and let it be turn'd round about the Axis A O.

I say, that a Body contain'd under conical Superficies
will be inscribed in the Spherical Segment.

This is proved as the foregoing *Lemma*.

Arch. Path. xv PROP. XVIII. Theorem.

Fig. 13.

LET the same Things be supposed which were in the first Lemma: and let the right Line (E) be drawn from the Extremity of the Diameter to the End of the Side next to the Diameter.

I Jay, that a Circle, the Square of whose Radius (I) is equal to the Rectangle $A E B$, contained under the Diameter $A E$, and the Subtense $E B$, is equal to all the conical Surfaces inscribed in the Sphere.

That is a Circle whose Radius (I) is a mean Proportion
betwixt A E and E B.

Because the right Lines BH , CG , DF , are equal to the right Lines BK , CM , DO , taken twice, (by 1. the Rectangle under one Side of the Figure inscribed in the greatest Circle, (to wit, under AB , or BC , or CD , or DE) and under all the joining Lines together, BH , CG , DF , is equal to the Rectangle under AB and BK , together with that which is under BC and the Compound of BC and CD .

CM, together with that which is under CD and the Compound of CM and DO, together with that which is under DE and DO; for so each of the Lines BK, CM and DO, are taken twice. But (by the 16th of this) the Rectangle under AB and all the joining Lines BH, CG, DF, taken together, is equal to the Rectangle AEB; that is, by the Hypothesis, to the Square of I. Therefore the Square of I is equal to the Rectangles under AB and BK, and under BC and the Compound of BK and CM, under CD and the Compound of CM and DO, and under DE and DO. Now let P be a mean Proportional betwixt AB and BK, and Q a Mean betwixt BC and the Compound of BK and CM; and R a Mean betwixt CD and the Compound of CM, DO; S, a Mean betwixt DE and DO. The Squares thereof of P, Q, R, S, (by 17. l. 6.) are equal to the abovesaid Rectangles. Wherefore seeing I have already shewed the Square of I to be equal to the same Rectangles, it must also be equal to the Squares of P, Q, R, S, together. Seeing therefore (by 2. l. 12.) Circles are betwixt themselves as the Squares of their Radius's; the Circle described of the Radius I, will also be equal to all the Circles together whose Radius's are P, Q, R, S, as is manifest from 1. l. 6. But the Circles of the Radius's P and S, are (by the 13th of this) equal to the conical Superficies which the Lines AB, ED, have produced; forasmuch as P is a mean Proportional betwixt AB the Side of the Cone, and BK the Radius of the Base; and S is a mean Proportional betwixt CD and DO; and the Circle of the Radius Q is (by the 13th of this) equal to that Segment of a conical Superficies which is intercepted betwixt the two parallel Circles of the Diameters CG, BH, because Q is a Mean betwixt BC and the Compound of BK, CM: And for the same Cause the Circle of the Radius R is equal to a Segment of a conical Surface, which is intercepted betwixt the parallel Circles of the Diameters CG, DF. Therefore the Circle described of the Radius I, is equal to the conical Surfaces inscribed in the Sphere taken all together. *Q. E. D.*

P R O P. XIX. Theorem.

LET the same Things be supposed which were in the second Lemma, and let the right Line AB be drawn from the Extremity of the Diameter (AE) to the End of AB, the Side next to the Diameter.

P 3

I say,

I say, that a Circle, whose Radius is a mean Proportional betwixt (EB) and (AO) the Axis of the Segment, is equal to all the conical Superficies inscribed in the spherical Segment DAF .

The Demonstration is altogether the same with that of the foregoing; only for *Proposition 16.* let *Proposition 17.* be cited.

PROP. XX. Theorem.

Fig. 15. CONICAL Superficies inscribed in a Sphere, at length end in the Superficies of the Sphere.

Let there be given a Superficies as small as you will, as X . It is manifest that within the spherical Superficies $ACEG$ there may be given some other Concentric thereto, which falls short of this by a Quantity less than X . Let $ACEG$ $DPLM$, be the the greatest Circle of both, as cut with a Plane through the Centre. Let there be drawn the Diameter ADE , and in D let NQ touch it. If the Arch AN be bisected in C , and again the Remainder be bisected, and so on, there will be left at last the Arch AB (as is manifest of itself) less than the Arch AN . If to this Arch the right Line AB be subtended, it is manifest, that it will not reach to the Circumference $PDM L$, and that it will be a Side of an Equilateral Figure of an even Number of Sides inscribed in the Circle $CAGE$, no Side whereof reacheth unto the Circumference $PDM L$. Wherefore if all be turn'd round about the Diameter AE , it is manifest that there will be inscribed in the exterior spherical Surface, conical Surfaces which include the spherical Surface, which is concentric to the other, and consequently, by *Axiom 3.* of this, are greater. Because therefore the spherical Surface $DPLM$ falls short of the spherical Surface $ACEG$, by a Quantity less than the given one X ; much more will the conical Surfaces be short of the said spherical Surface $ACEG$ by a Quantity less than the given one X , and (by *Defin. 6 l. 12.*) consequently will end in the Superficies $ACEG$. *Q. E. D.*

PROP. XXI. Theorem.

Fig. 17. CONICAL Superficies inscribed in a spherical Segment DAF , end in the spherical superficies of the Segment itself.

This may be demonstrated by the same Reasoning as the foregoing was.

PROP. XXII. Theorem.

It was demonstrated, Prop. 18. that a Circle *Fig. 16.* whose Radius is a mean Proportional betwixt the Diameter AE , and the right Line EB , which is drawn from the Extremity of the Diameter unto the End of the Side AB , next to the Diameter, is equal to all the conic Superficies inscribed in the Sphere.

I say, that this Circle (see Def. 6. l. 12.) ends at length in a Circle, whose Radius is AE , the Diameter of the Sphere.

For if more and more Sides be infinitely inscribed in the greatest Circle (which then being turn'd round about AE , produce conical Superficies) it is manifest, that the Side AB becomes at length less than any given right Line, and consequently that the Subtense EB approaches to the Diameter AE within a Distance less also than any given one; from whence it comes to pass that the Difference of those AE, BE , becomes likewise less than any given one. Therefore much more shall the mean Proportional betwixt AE, BE , which is always greater than BE , differ from AE at length by a Defect less than any given one. Therefore the Circle also whose Semi-diameter is a Mean betwixt AE and BE , will at length differ from a Circle whose Semi-diameter is AE , by a Defect less than any given one whatsoever, that is, will end (Def. 6. l. 12.) in it. *Q. E. D.*

This, which is clear enough of itself, there is no need to demonstrate more operosely.

PROP. XXIII. Theorem.

It was demonstrated, Prop. 19. that a Circle, *Fig. 17.* whose Radius is a mean Proportional betwixt EB and the Axis of the Segment AO , is equal to all the conic Superficies inscribed in the spherical Portion DAF .

P 4

I say,

I say, that this Circle ends in a Circle, whose Radius is the right Line AD , drawn from the Vertex of the Segment unto the Circumference of the Circle $DQFN$, which is the Base of the Segment.

For because it now appears from the foregoing Demonstration that EB doth at length end in AE ; it will also be manifest that the mean Proportional betwixt EB and AO doth at length end in the mean Proportional betwixt AE and AO , that is, (by *Coroll. 2. p. 8. l. 6.*) in AD itself. It is therefore manifest that the Circle also whose Radius is a mean Proportional betwixt EB and AO , doth end in the Circle of the Radius AD . $Q.E.D.$

A Lemma to the following Proposition.

IF the Diameter of one Circle be double to the Diameter of another, the one Circle will be four-fold to the other. This is manifest from *Prop. 2. l. 12.* and *Defin. 10. l. 1.*

Archimedes's XXXV R O P. XXIV. Theorem.

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Fig. 16.

THE Superficies of every Sphere is four-fold to the greatest Circle of the same Sphere.

This most noble Theorem of *Archimedes*, we shall, from what goes before, expeditiously demonstrate in this manner.

Let an Ordinate Figure, the Sides whereof are measured by the number Four, be understood to be inscribed in the greatest Circle of a Sphere about the Diameter AE . Let this Figure be turn'd round about AE , and so produce conical Surfaces inscribed in the spherical Surface, and let EB be drawn. It hath been demonstrated above (18. of this) that all conical Surfaces inscribed in a Sphere are equal to the Circle, the Square of the Radius whereof is equal to the Rectangle AEB , that is, whose Radius is a mean Proportional betwixt AE and EB . And this will always happen although the Inscription be infinitely continued. Wherefore seeing the inscrib'd conical Surfaces, (by 20. of this) will at length end in the spherical Surface, and the Circle whose Radius is a Mean betwixt AE and EB , will at length end (by 22. of this) in the Circle whose Radius is AE ; the spherical Surface itself also (by 2. of this) will be equal

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to the Circle of the Radius A E, that is, by the foregoing Lemma, to four times the greatest Circle A C E G. Q.E.D.

He that shall read *Archimedes*, will find that the Way here used in demonstrating this most noble Theorem, is much shorter and clearer than that of *Archimedes*.

Corollary.

FROM this admirable Theorem, whereby *Archimedes* hath purchas'd to himself an immortal Name amongst the Geometricians, a Circle equal to a spherical Surface is obtain'd; that, to wit, whose Semi-diameter is the Diameter of a Sphere, or whose Diameter is double to the Sphere's Diameter.

Scholium.

WE are now well provided for the measuring of a spherical Surface, the chief amongst all Curve ones. And it is perform'd these two Ways.

1. Let the greatest Circle of the Sphere be measured, (according to *Schol. Prop. 6.* of this) and let it be multiplied by 4. As, if the greatest Circle of the Orb of the Earth be found to contain 49,075,500 square Miles, or more exactly, 49,081,250 square Miles, then, according to this, 196,325,000 square Miles are contain'd in the whole spherical Surface.

2. The Diameter of a Sphere multiplied by the Circumference of the greatest Circle gives you the spherical Superficies. According to which, if the Earth's Diameter consist of 7,853 Miles, and consequently the Circumference of the greatest Circle consists of 25,000, the whole spherical Surface will be in the same Miles 196,325,000; for $7,853 \times 25,000 = 196,325,000$.

The Demonstration appears from *Corol. 1. Prop. 5.* of this; for a Rectangle under the Diameter of a Sphere, and the Circumference of the greatest Circle, is according to that *Corollary* four-fold of the greatest Circle.

PROP. XXV. Theorem.

THE Surface of any spherical Portion whatever (as D A F) is equal to a Circle, whose Radius is the right Line (A D) drawn from the Vertex of the

XLIX

Arch. Prop. XXXVIII XLIX

Fig. 17. 31

the Portion to the Circumference of the Circle (D F N) which is the Basis of the Segment.

Let a Figure, Equilateral and of an even Number of Sides, the Base being set aside, be imagin'd to be inscrib'd in the Section of the greatest Portion about the Axis A O; this Figure being turn'd round about A O, will inscribe conical Surfaces in the Portion. Let the right Line E B be drawn also as above (in 18 and 19 of this.) All the conical Surfaces now inscribed are equal (by the 19th of this) to the Circle whose Radius is a mean Proportional betwixt E B and the Axis of the Segment A O. And this will always happen if the Inscription be infinitely continued. Wherefore seeing both the conical Surfaces inscrib'd in the Segment end at length (by 21. of this) in the spherical Surface of the Segment, and the Circle whose Radius is a Mean betwixt E B and A O, ends (by 23.) in the Circle of the Radius A D; the spherical Surface of the Portion also D A F (by 2. will be equal to the Circle of the Radius A D. Q. E. D.

This is another of the more noble Inventions of *Archimedes*, which, as the former, we have demonstrated in a much shorter and clearer Way, than he did.

PROP. XXVI. Theorem.

Fig. 18.

THE Superficies of a right Cylinder circumscrib'd about the Sphere (as the Cylinder H P S V) is equal to the Surface of the Sphere.

And if a Cylinder and Sphere be cut by Planes perpendicular to the Axis (B G;) each Segment of the cylindrical Surface will be equal to each Segment of the spherical Surface.

Part I. Because the Side H P of the Cylinder is (by the Hypothesis) equal to P S, the Diameter of the Base; the cylindrical Surface H S will be (by *Coroll. p. 12.* of this) four-fold of the Base; that is, of the greatest Circle of the Sphere inscrib'd in the Cylinder; of which, seeing (by the 24th of this) the spherical Surface it self is also four-fold, this will be equal to the cylindrical Surface, Q. E. D.

Part II. Let the right Lines B O, G O, be drawn. Because the Angle B O G (by 31. l. 3.) is right, as being the

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Angle in the Semi-circle, and OC falls perpendicular from it upon BG ; BO (by *Coroll. 2. p. 8. l. 6.*) will be a mean Proportional betwixt GB and BC , that is, betwixt IT and HI . Therefore the Circle of the Radius BO (by 11. of this) will be equal to the cylindrical Surface HT . But the same Circle is also (by the foregoing) equal to the Segment of the spherical Surface OBK . Therefore the cylindrical Surface HT , and the spherical OBK , are equal.

Then because it is shew'd in the same manner that the cylindrical Surface HX is equal to the spherical QBR , the remaining cylindrical Surface IX will be equal to the remaining spherical Surface $QOKR$, which is intercepted betwixt two parallel Circles.

And from these the Proposition is manifest of all Segments.

[Corollary. "Hence the Superficies of a Cylinder circumscrib'd about a Circle is double to the Bases.]

PROP. XXVII. Theorem.

THE Segments of a spherical Surface divided by parallel Circles have that Proportion amongst themselves, which the Segments (BC , CD , DA , AE , EF , FG) of that Diameter (BG) which is perpendicular to the parallel Circles have amongst themselves. Fig. 18.

It follows from the foregoing. For by that the Segments of the spherical Surface OBK , $QOKR$, $MQRN$, &c. are equal to the cylindrical HT , IX , LN , &c. But these (by 13. l. 12.) have the same Proportion betwixt themselves, which the Segments of the Axis BC , CD , DA , &c. have. Therefore those also have the same Proportion. Q. E. D.

Scholium.

FROM hence the Proportion of Zones and Climates betwixt themselves becomes known. For they are to one another as the Segments of the Axis, which are known from the Table of Sines. Fig. 19.

From the same also we learn to measure the Segments of a spherical Surface. For because both the whole Surface of the Sphere is known from *Schol. Prop. 24.* and the Proportion

tion of the Segments, the same as that of the Parts of the Axis, is also given; it is manifest that each of the Segments become known.

Now both the foregoing, and all the rest of the Theorems which follow, are altogether singular and admirable, and well worthy that those who are studious of Geometry should give all Diligence to understand them.

A Lemma to the following.

Fig. 19.

IF a Plane (Q N) touch a Sphere in (O) a right Line (A O) from the Center to the Contact, is perpendicular to the Plane.

Let Q N, the touching Plane and the Sphere, be cut through the Contact with two Planes, which in the Sphere may produce the Circles O G, O D, but in the Plane Q N, the right Lines C O, I O, which shall touch the Circles in O. Therefore by 18. l. 3. A O is perpendicular to both I O and C O, and consequently by 4. l. 11. perpendicular to the Plane Q N. Q. E. D.

Arch Prop XXXV P R O P. XXVIII. Theorem.

Fig. 20, 22.
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EVERY Sphere is equal to a Cone (Z O) whose Altitude (K O) is equal to the Radius of the Sphere; and the Base (Z) equal to the Superficies of the Sphere.

Let some Polyedral Body be understood to be circumscribed about the Sphere, and let the solid Angles thereof be cut off by new Planes touching the Sphere. Which being done, there will arise another Polyedral Body containing the Sphere, but less than the former, and consisting of more Angles, and having a Surface compounded of more Tangent Planes in Number, but less in Magnitude. If the solid Angle of this Polyedrum be again cut off by new Tangent Planes, and the Angles of the third Polyedrum thence arising likewise, and so on for ever; it will come to pass at length, that both the Polyedrum will exceed the Sphere by a Solid less than any given one whatsoever; and the Surface thereof compounded of Tangent Planes (which, as I said, are endlessly less in Magnitude, and more and more in Number than they were before) will exceed the spherical Surface also by a

Plane

Plane less than any given one whatever. Both which Things although they might be demonstrated, yet because they are of themselves manifest enough, I shall for Brevity-sake take for granted. These Things being thus stated, we proceed.

The Polyedrum now defin'd is compounded of Pyramids, the common Top whereof is the Center of the Sphere, and the Bases are Tangent Planes, which constitute the Surface of the Polyedrum. And because the right Lines drawn from the Center A unto the Contacts of each of the Planes, are (by the foregoing *Lemma*) perpendicular to each of the Planes; therefore the Height of all the Pyramids, whereof the Polyedrum consists, will be equal; to wit, AB the Radius of the Sphere. If therefore the Plane X be supposed equal to the Surface of the Polyedrum itself, and upon it there be erected a Pyramid at the Height MN, which is also equal to the Radius of the Sphere AB; it is manifest (by 6. *l.* 12.) that all the abovesaid Pyramids, that is, the whole Polyedrum are equal to the Pyramid XN. After the same manner all the rest of the Polyedrums containing the Sphere, which from the perpetual Abscission of the solid Angles will arise one after another infinitely, are always equal to the Pyramids (represented by XN) the Altitudes whereof MN are the Radius of the Sphere; but the Bases (X) equal to the Surfaces of Polyedrums encompassing the Sphere. Wherefore seeing at length both the Polyedrums (as I said above) do end in a Sphere, and the Pyramids (XN) as I will shew by and by, do end in the Cone ZO; the Sphere also (by 1. of this) will be equal to the Cone. Q. E. D.

But that the Pyramids XN end in a Cone I thus shew: The Surfaces of Polyedrums end in the Surface of the Sphere, as it was taken for granted above. But the Bases X of the Pyramids XN, are always supposed equal to the Surfaces of the Polyedrums; and Z the Base of the Cone, ZO is, by the Hypothesis, equal to the Surface of the Sphere; therefore the Bases X also will end in the Base Z; and consequently seeing the Pyramids XN to be to the Cone, which, by the Hypothesis, is of equal Height (by *Coroll. Prop. 11. l.* 12.) as the Base X is to the Base Z, the Pyramids also will end in the Cone.

The Demonstration of this Proposition and the following, is altogether diverse from that which *Archimedes* made use of, which indeed is very subtil and ingenious, but prolix and difficult;

difficult ; to which there are premised two Positions that are manifest, and eleven Propositions, besides others, not a few, on which they depend. But the Theorem itself, as propounded by *Archimedes*, is thus ; Every Sphere is fourfold of a Cone, which hath a Base equal to the greatest Circle of the Sphere, and its Altitude equal to the Radius.

Scholium.

FROM this noble Theorem is deduced the Mensuration of the most noble of solid Figures. For if the sixth Part of the Diameter, or the third Part of the Semi-diameter be multiplied by the Surface of the Sphere, already known by *Schol. Prop. 24.* there will arise the Solidity of the Sphere.

Suppose the Superficies of the Earth be found to contain 196,325,000 square Miles, and let the third Part of the Semi-diameter consist of 1309 such Miles. Multiply the two Numbers together, the Product 256,989,425,000 will be the Number of the cubic Miles of the Earth's Solidity.

For seeing a Sphere (by this *Prop.*) is equal to a Cone whose Altitude is the Radius of the Sphere, and its Base the Surface of the same Sphere, and the Solidity of the Cone (by *Schol. Prop. 6.* of this) is produced from the third Part of the Altitude (that is, of the Radius of the Sphere) multiplied by the Base (that is, the Surface of the Sphere) the Sphere's Solidity also is obtain'd from the third Part of the Radius multiplied into the Superficies.

Arch. Prop. **PROP. XXIX. Theorem.**

Fig. 23. **EVERY** Sector of the Sphere is equal to a Cone, whose Altitude is the Radius of the Sphere, and the Base the spherical Superficies of the Sector.

First, let the Sector A E C G be less than an Hemisphere. Let a right-lin'd polyedral Body be understood to be circumscrib'd about the Sector. Now, if all the remaining Ratiocination be carried on after the same manner as was done in the foregoing, the Thing sought will be concluded in the same manner. This Thing alone will require to be shew'd, upon which indeed the whole Reasoning depends ; to wit, that the Superficies of the Polyedrum, which is compounded

Planes on every Side, touching the Surface of the Sphere ECG , is greater than the Surface ECG . Which is done thus. Let another equal and like Surface be conceiv'd to be set to the Surface ECG encompass'd with touching Planes in the very same manner as the other is. There will now (by *Axiom* 3. of this) the whole Surface compounded of Planes, be greater than the whole spherical Surface. Therefore half the Surface compounded of Planes will also be greater than half the spherical Surface ECG .

Then let the Sector $AEBG$, be greater than an Hemisphere. Both Sectors taken together, are (by the foregoing) equal to a Cone whose Height is the Radius of the Sphere, and its Bases the whole Superficies; that is, (by *11. l. 12.*) to two Cones, which have the same Height, but have their Bases equal to the Segments of the spherical Superficies ECG , EBG . But one of the Sectors $AECG$, that which is less than an Hemisphere, is by Part I. equal to a Cone, whose Altitude is the Radius, and its Base the Surface ECG . Therefore the other Sector $EABG$, is equal to the other Cone whose Height is the Radius, and its Base the remaining spherical Surface EBG . *Q. E. D.*

Corollary.

SEEING (by 25. of this) the Superficies ECG is equal to the Circle of the Radius CG , and the Superficies EBG equal to the Circle of the Radius BG ; the Sectors $AECG$, and $AEBG$, will be equal to Cones, whose Altitude is the Radius of the Sphere, and their Bases, Circles of the Radius's CG and BG .

Scholium.

FROM these Things is deduced the measuring both of *Fig. 23.* Sectors and Segments of Spheres; of Sectors (as appears from *Schol. Prop. 6.* of this) if the third Part of the Radius be multiplied by the spherical Surface of the Sectors, which is already known from *Schol. Prop. 27.* or by the Circle of the Radius CG or BG ; and of Segments, if the Cone EAG be measured, and be taken away from the Sector, if it be less than an Hemisphere; but added thereto, if it be greater.

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Fig. 18. The Segment (M Q R N) which lies betwixt two Circles whether parallel or not parallel, is measured; if the Segments Q B R and M B N already known, be subtracted one out of the other.

PROP. XXX. Theorem.

Fig. 24. *A* Hemisphere (E O B D) is double to the Cone (E B D) which hath the same Base and Altitude with itself.

The Cone, whose Basis is the Hemispherical Superficie E O B D, and its Altitude the Radius A B, is to the Cone E B D (by 11. l. 12.) as Base is to Base; that is, as the hemispherical Surface E O B D, is to the greatest Circle P T. Therefore seeing the hemispherical Superficies E O B D double to the greatest Circle (by 24 of this) the Cone also which hath the Superficies E O B D for its Base, and the Radius A B for its Altitude, is double to the Cone E B D. But (by 28. of this) the Hemisphere is equal to a Cone which hath the Radius for its Altitude, and the hemispherical Superficies for its Base. Therefore the Hemisphere also double to the Cone E B D. Q. E. D.

Archimedes III PROP. XXXI. Theorem.

Fig. 25. *L*ET a Sphere be divided into two Segments I L B G, I S K G, by the Plane I Q G which doth not pass through the Center A; and let the Diameter B O K be perpendicular to the cutting Plane.

As the Altitude O B of the Segment, I L B G, is to the Radius of the Sphere A B; so let O K, the Altitude of the other Segment, be made to the other Line K N.

In like manner, as O K, the Altitude of the Segment I S K G, is to the Radius A K or A B; so let the Altitude O B of the other Segment be made to the other Line B D. Which Thing being supposed, I say,

1. T

1. The Cones ING and IDG whose Altitudes are ON , OD , and IQT , their common Base, are equal to the spherical Segments.

2. There is the same Proportion of the Segments as there is of the right Lines DO , NO .

3. The Segment $ISKG$ is to the greatest Cone IKG inscribed in it, as NO is to KO ; and the Segment $ILBG$ is to the greatest Cone IBG inscribed in it, as DO is to BO .

Part I. Let the Sphere and Cones be cut by a Plane through the Diameter BK . There will be produced in the Sphere the greatest Circle $BLKG$, and in the Cones the Triangles BIG , IKG . And because BOK , the Diameter is (by the Hypothesis) perpendicular to the Circle QT , DOB (by Def. 3. l. 11.) will be a right Angle. The Angle in the Semi-circle is also a right one (by 31. l. 3.) because therefore in the Triangle BIK , there is drawn from the right Angle, IO perpendicular to the Base BK ; BI will be to IO , as (by 8. l. 6.) BK to KI . Therefore the duplicate Proportion of BI to IO is equal to the duplicate Proportion of BK to KI ; that is, (because BK , KI , IO by Coroll. 2. p. 8. l. 6. are three Proportionals) equal to the Proportion of BK to KO .

Then because OB is (by the Hypothesis) to BD , as OK is to the Radius AB ; by Inversion it will be always as, DB is to BO , as AB to OK ; and by Permutation as, DB is to BA , as BO to OK ; and by compounding as, DA is to BA , as BK is to OK . Because therefore we have already shew'd the Proportion of BK to OK , to be duplicate to the Proportion of BI to IO , and consequently (by 2. l. 12.) equal to the Proportion betwixt the Circles describ'd by the Radius's BI , IO ; DA will also be to BA , as the Circle of the Radius BI , to the Circle of the Radius IO . Therefore the Cone under the Altitude DA , and for the Base, the Circle of the Radius IO ; that is, the Circle QT , is equal to the Cone under the Altitude BA (by 15. l. 12.) which hath for its Base the Circle of the Radius BI ; that is, (by Coroll. 29. of this) the spherical Sector $AIBG$. Wherefore if the same Cone IAG be added, as well to the Sector $AIBG$, as to the Cone under DA , and the Circle QT , the Wholes will be equal, to wit,

Q

the

the spherical Segment $ILBG$, will be equal to two Cones, whereof one is that which is under the Base QT and the Altitude DA , and the other IAG is under the same Base QT , and the Altitude OA . But these two Cones (by *14. l. 12.*) make up the Cone IDG . Therefore the Segment $ILBG$ will be equal to the Cone IDG . *Q. E. D.*

By the same Reasoning, the Segment $ISKG$ will be equal to the Cone ING , with this only Change, that the Cone IAG , which before was added, be now taken away.

Part II. This is manifest out of the first. For the Cones IDG and ING are betwixt themselves (by *p. 14. l. 12.*) as are DO and NO . Therefore the Segments also $ILBG$, $ISKG$, equal to those Cones, are betwixt themselves, as the right Lines DO , NO .

Part III. This likewise is manifest from the first. For the Cone IDG is to the Cone IBG , (by the same) as DO is to BO . Therefore the Segment also $ILBG$, which is equal to the Cone IDG , is to the Cone IBG , as DO is to BO .

Scholium.

FROM the first Part of this Proposition there arises another Way of Measuring spherical Segments, and that a very easy one; if, to wit, the Cones IDG , ING , be measured; which will be done, if the third Parts of the right Lines DO , NO be drawn into the Circle QT .

Arch. Prop. II? PROP. XXXII. Theorem.

Fig. 24.

A Right Cylinder (GK) is both in Solidity and the whole Superficies to the Sphere about which it is circumscrib'd, as 3 to 2.

Let BQ be the common Axis of the Sphere and Cylinder, and EBD the greatest Cone inscrib'd in the Hemisphere $EOBD$. Because the Cylinder EK (half of GK) is (by *10. l. 12.*) triple to the Cone EBD , while the Hemisphere is double to the same Cone, (by *30. of this*) it is manifest that the Cylinder EK is to the Hemisphere as 3 to 2. Therefore also the whole Cylinder GK , is to the whole Sphere $QEBD$, as 3 to 2. Which was the first Part.

Then because the Side of the Cylinder KN is equal to GN the Diameter of the Base, its Superficies without the Bases will be four-fold (by *Coroll. p. 12. of this*) of the

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XXXVIII

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Base M I, and consequently taken together with the Bases, that is, the whole Superficies of the Cylinder, will be six-fold of the Base M I, which is equal to the greatest Circle of the Sphere. But the Superficies of the Sphere is four-fold of that greatest Circle. Therefore the whole Superficies of the Cylinder G K is to the Superficies of the Sphere, as 6 to 4, or as 3 to 2. Which was the other Part.

Therefore a Cylinder is both in Solidity and the whole Superficies to the Sphere, about which it is circumscrib'd, as 3 to 2. Q. E. D.

Scholium.

It is an Argument what a great Value *Archimedes* puts upon this Theorem, that he would have a Sphere inscrib'd in a Cylinder set upon his Tomb. And perhaps amongst many other famous Discoveries, this chiefly and above all others pleas'd him for this Reason, to wit, because there was one and the same rational Proportion both of Bodies, and of the Surfaces which contain them. We have demonstrated a like Identity of Affection betwixt Rings, and the Surfaces of Rings, in the Fourth Book of our *Cylindicks and Annularies*, *Prop.* 13, 14, 15. And another famous Example of the same hath also offer'd itself to me in the Sphere itself. For I have found, that like as a Sphere is to a right Cylinder which encompasseth it (which will necessarily be Equilateral) as 2 is to 3, and this both in respect of Solidity and Surface; so likewise the Sphere hath to an Equilateral Cone encompassing it, that Proportion which 4 hath to 9; and this both in regard of Solidity and Superficies. From which this also follows, That the sesquialteral Proportion found by *Archimedes* in the Sphere and Cylinder, is contained in three Solids, Sphere, Cylinder and Equilateral Cone. The Demonstration of both which things, with some other Theorems of my own, in which the wonderful Nature of the Sphere will more appear, I shall subjoin in the thirteen following Propositions.

PROP. XXXIII. Theorem.

THE Superficies of a Sphere is double to the Superficies of a square Cylinder inscrib'd in the Sphere. Fig. 26.

Let A K L be the Square inscrib'd in the greatest Circle of a Sphere, from which turn'd round, there is described a

Q 2

square

square Cylinder; and let AL be drawn as a Diameter common to the Cylinder and Sphere. Because the Square AL is (by 47. *l.* 1.) equal to the equal Squares AK , KL , it will be double to one AK . Therefore also the Circle of the Diameter AL , is (by 2. *l.* 12.) double to the Circle whose Diameter is AK ; to wit, to the Circle CN . But the Superficies of the Sphere is (by 24. of this) four-fold to the Circle whose Diameter is AL ; for that is the greatest Circle of the Sphere, seeing AL is the Diameter of the Sphere. Therefore the Superficies of the Sphere is eight-fold of the Circle CN . But because LK , KA (by the Hypothesis) are equal, the cylindrical Superficies ACL is (by *Coroll.* p. 12. of this) quadruple of the Circle CN . Therefore since the Superficies of the Sphere is eight-fold of the same Circle it will be double to the cylindrical Superficies. *Q. E. D.*

PROP. XXXIV. Theorem.

Fig. 26.

THE Superficies of a Sphere hath that Proportion to the whole Superficies of a square Cylinder inscrib'd in it, which 4 hath to 3.

Let the same Things be suppos'd which were in the foregoing Demonstration. Because, by the Hypothesis, LS the Side of the Cylinder, and AK the Diameter of the Base thereof, are equal, the cylindrical Superficies CL will be quadruple (by *Coroll.* p. 12. of this) to the Base CN and consequently the whole Superficies of the Cylinder is to both Bases CN and SL , as 6 is to 2. But the Superficies of the Sphere is to both Bases together, CN , SL , as 8 to 2, seeing in the foregoing it was shew'd that it is to one Base as 8 to 1. Therefore the Superficies of the Sphere is to the cylindrical Superficies CL , as 8 is to 6, or 4 to 3. *Q. E. D.*

Corollary.

THE whole Superficies of a right Cylinder describ'd about a Sphere, is to the whole Superficies of an Equilateral Cylinder inscrib'd, as 2 is to 1. For the circumference of the Cylinder describ'd is to the circumference of the inscrib'd, as 2 is to 1. (by 32. of this.) But the Spheric Superficies is to the circumference of the inscrib'd, as 8 is to 6, or 4 to 3, by this present Proposition. Therefore the Circumscrib'd is to the Inscrib'd, as 12 is to 6, or 2 to 1.

PRO

PROP. XXXV. Theorem.

A Portion of any spherical Superficies whatever (as Fig. 26. *ILBG*) hath the same Proportion to the Superficies of the greatest inscribed Cone, which (*BG*) the Side of the Cone hath to (*GO*) the Radius of the Base.

Because (by 25. of this) the Superficies of the Portion *ILBG* is equal to the Circle of the Radius *BG*; the Proportion thereof to *QT*, that is, to the Base of itself and of the Cone, will be duplicate to the Proportion of the conical Superficies *IBG* to the same Base *QT*. Therefore it is manifest, (by Definition 10. l. 5) that the Superficies *ILBG* is to the conical Superficies *IBG*, as the same conical Superficies *IBG*, is to the Base *QT*. Wherefore being the conical Superficies *IBG*, is the Base *QT*, as *ILBG* (by 14 of this) is to *GO*, the Superficies of the Portion will also be to the conical Superficies *IBG* inscrib'd in, as *BG* is to *GO*. Q. E. D.

PROP. XXXVI. Theorem.

THE Superficies of the Hemisphere (*EOBD*) Fig. 24. hath that Proportion to *EBD* the Superficies of the greatest right inscribed Cone, which in a Square the Diameter hath to a Side; and that Proportion to the Superficies of a like Cone circumscribed, as the Side in a Square hath to the Diameter.

I. The Demonstration of the first Part is manifest from Fig. 6. l. 4.

For *EOBD* the Superficies of any Portion whatever, and consequently of the Hemisphere, is to the conical Superficies inscrib'd, as *BD* is to *DA*. But *BADK* is a Square whose Diameter is *BD*, and the Side *DA*.

Part II. Let *EBG* be half of the Square circumscrib'd about the Circle, (whose Center is *O*) which *EBG* being divid'd about the Axis *AB*, let there from thence be projected a Cone circumscribed about the Hemisphere. Now, because the Square *EC* is (by 47. l. 1.) double to the Square *EBG*, or *GI*, the Circle of the Diameter *EC* also is (by 2.

Q 3

l. 12.)

l. 12.) double to the Circle whose Diameter is GI , that is, to the Circle $HGD I$. But (by 24 of this) the Superficies of the Hemisphere included in the Cone $EB C$, is double to the same Circle. Therefore the Circle of the Diameter EC is equal to the hemispherical Surface. Wherefore seeing the conical Superficies $EB C$ is (by 14. of this) to the Circle of the Diameter EC , to wit, to its own Base, as the Side BE is to EO , the Radius of the Base; it will be also to the hemispherical Superficies inscribed in it, as BE is to EO , that is, as the Diameter in a Square is to a Side. *Q. E. D.*

PROP. XXXVII. Theorem.

The same Figure with Fig. 13. l. 5.

A Sphere hath the same Proportion to a square conical Rhombus circumscribed about it, both in respect of the Solidity and Surface, which in a Square the Side hath to the Diameter.

Let the Square $EB C F$ be circumscrib'd about $HGD I$ the greatest Circle of a Sphere, from which Square, if turn'd round about the Axis BF , let a conical Rhombus encompassing the Sphere be produced.

As EB , a Side of the Square (see *Fig. 6. l. 4.*) is to the Diameter EC , even so let S be made to R ; (see *Fig. 13. l. 5.*) and let this Proportion be continued through four Terms S, R, Q, O , the Proportion then of S to O will be triplicate to the Proportion of S to R ; that is, of EB to EC and the Proportion of O to R will be duplicate to the Proportion of O to Q , or of R to S ; that is, of EC to EB and consequently (by 20. *l. 6.*) O is to R as the Square of EC is to EB ; from whence (by *Schol. Prop. 6. and 7. l. 4.*) O is double to R . These things being thus settled, let the Sphere $EB C F$ be understood to be circumscribed about the conical Rhombus. Thus the Sphere $HGD I$ will be to the Sphere $EB C F$ (by 18. *l. 12.*) in the triplicate Proportion of the Diameter GI or EB to the Diameter EC ; that is, (as I have already shew'd) it will be as S to O . But the Sphere $EB C F$ is to the conical Rhombus inscribed in it (by 30 of this) as 2 is to 1; that is, (as I have shew'd above) as O is to R . Therefore by Equality of Proportion, the Sphere $HGD I$ is to the same Rhombus which is described about it, as S is to R ; that is, as in a Square the Side is to the Diameter EC . Which was the first Part. The second Part of the foregoing, it appears, that the

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Superficies of the Hemisphere is to the Superficies of the Cone EBC , and consequently the Superficies of the whole Sphere is to the Superficies of the whole Rhombus $EBCF$, as in a Square the Side is to the Diameter. Therefore the Sphere, as well in Solidity as in the Superficies, is to the Square Rhombus $EBCF$, as in a Square the Side is to the Diameter. *Q. E. D.*

PROP. XXXVIII. Theorem.

THE Superficies of the Portion ($BGKD$) which *Fig. 27.* contains an Equilateral Cone (BKD) is double to the Superficies of the same Cone.

This is manifest from 35. For the Superficies of the Portion $BGKD$ is to the inscrib'd conic Superficies (by 35. of this) as BK is to BA . But because the Cone BKD is suppos'd to be Equilateral, KB is equal to BD , and consequently double to BA . Therefore the Superficies $BGKD$ is also double to the inscribed conical Superficies. *Q. E. D.*

PROP. XXXIX. Theorem.

THE Superficies of a Sphere is to the whole *Fig. 27.* Superficies of an Equilateral Cone inscribed in it, as 16 to 9.

Let Z be the Center of the Sphere, and BKD the Equilateral Cone inscribed, and $KZAO$ the Axis common to the Sphere and Cone. If the Sphere and Cone be cut through this, there will be produced in the Sphere the greatest Circle, and in the Cone, the Equilateral Triangle BKD , one Side whereof will be the Diameter of the Basis of the Cone QT . And because the Axis of the Cone KA is perpendicular to the Base QT , BAK (*Def. 3. l. 11.*) will be a right Angle. Therefore the Square of BA is equal to the Rectangle KAO . (*Coroll. 1. p. 17. l. 6.*) Now because the Side of the Equilateral Triangle cuts off (*Coroll. p. 15 l. 4.*) a fourth Part of the Axis AO , the Rectangle KAO , that is, the Square of BA , will be triple to the Square of AO (by 1. l. 6.) Wherefore seeing the Square of the Radius ZO is (*Coroll. 3. p. 1. l. 2.*) quadruple of the Square of AO , the Square of the Radius ZO will be to the Square of the Radius BA , as 4 is to 3.

Q. 4

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Therefore the Circle O B K D is also (by 2. l. 11.) to the Circle Q T, as 4 is to 3. Therefore four Circles, O B K D that is, (by 24. of this) the whole spherical Superficies D G is to the Circle Q T, as 16 is to 3. But (*Coroll. p. 14. of this*) the Superficies of the Equilateral Cone B K D is to the Circle Q T, to wit, its own Base, as 2 is to 1; and consequently the whole Superficies of the Cone B K D that is, including its Base, is to the Base, to wit, the Circle Q T, as 3 is to 1, or 9 to 3. Therefore, seeing I have shew'd that the Superficies of a Sphere is to the same Circle as 16 is to 3, the Superficies also of the Sphere D G will be to the whole Superficies of the Equilateral Cone, as 16 is to 9. *Q. E. D.*

Or otherwise thus.

BEcause, (by *Coroll. 5. Pr. 15. l. 4.*) the Side B D of the Equilateral Triangle cuts off a fourth Part of the Axis A O, the spherical Superficies B O D will be a fourth Part by 27. of this, and consequently the Superficies D G D R three fourth Parts of the Superficies of the whole Sphere. Wherefore if the whole Superficies be suppos'd to be 16, the Superficies B G K D will be 12. But (by the foregoing) the Superficies B G K D is double to the conical Superficies B K D, and consequently is to it, as 12 to 6. Therefore the whole Superficies of the Sphere is to the conical B K D as 16 is to 6. Then because the Superficies of the Cone B K D (as being Equilateral) is (by *Corol. 1. Pr. 14. of this*) double to the Base Q T, it is manifest, that the conical Superficies B K D (to wit, without the Base) is to the whole Superficies of the Cone, as 2 is to 3; that is, as 6 to 9. Therefore by Equality of Proportion, the whole Superficies of the Sphere is to the whole Superficies of the Equilateral Cone inscrib'd, as 16 to 9. *Q. E. D.*

P R O P. XL. Theorem.

Fig. 28. **T**HE Superficies of the Sphere bears the Proportion to the whole Superficies of an Equilateral Cone circumscrib'd about it, that 4 doth to 9.

Let there be circumscrib'd about the greatest Circle of Sphere B P M, the Equilateral Triangle D O F, by which turn'd round about the Axis O A B, let there be produced an Equilateral Cone circumscribed about the Sphere. *And*

let there also be a Circle D O F, tangent to the Sphere at O, and concentric with the Circle B P M, reduced to N. *As is manifest* B K. Wherefore the Superficies of the Cone B K D is to the Circle B K D, as 2 is to 1. But it hath also been shew'd, that the Superficies of the Cone B K D is to the Circle B K D, as 2 is to 1. Therefore the Superficies of the Cone B K D is to the Circle B K D, as 2 is to 1. Therefore the Superficies of the Cone B K D is to the Circle B K D, as 2 is to 1.

Circle B P M, the whole Surface of this) triple the Superficies of the Cone is to the Superficies of the Sphere. (*this*) of the Sphere is to the Superficies of the Equilateral Cone to which it is inscrib'd. *Corollary.*

that the Area of the Sphere is to the Area of the Equilateral Cone circumscribed about it, as 4 to 9.

2. " That the Area of the Sphere is to the Area of the Equilateral Cone circumscribed about it, as 4 to 9.

3. " That the Area of the Sphere is to the Area of the Equilateral Cone circumscribed about it, as 4 to 9.

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THE whole Superficies of the Sphere is to the whole Superficies of the Equilateral Cone circumscribed about it, as 4 to 9.

By the foregoing Theorem, the Area of the Sphere is to the Area of the Equilateral Cone circumscribed about it, as 4 to 9.

let there also be circumscribed about the Equilateral Triangle DOF , the Circle $N D L O F$, which, as is manifest, is concentric to the former; and let the Axis $O A B$ be produced to N . Because BN is a fourth Part of the Axis ON , as is manifest from (*Corol. 5. Pr. 15. l. 4.*) ON is double to BK . Wherefore the Proportion betwixt Circles being duplicate (by 2. l. 12.) of the Proportion of the Diameters, the Circle BPM will be to the Circle $N D L O F$, as 1 to 4. But it hath already been shew'd in the foregoing Demonstration, that the Circle $N D L O F$, is to the Circle QT , the Base of the Equilateral Cone inscrib'd in the Sphere FL , as 4 is to 3. Therefore, by Equality of Proportion, the Circle BPM as to the Circle QT , as 1 is to 3. But the whole Surface of the Cone DOF is (by *Corol. 1. Pr. 14.* of this) triple to QT . Therefore the whole Superficies of the Cone is nine-fold of the Circle BPM . Wherefore seeing the Superficies of the Sphere TP is quadruple (by 24. of this) of the same Circle BPM , the whole Superficies of the Equilateral Cone DOF is to the Superficies of the Sphere to which it is circumscrib'd, as 9 is to 4. *Q. E. D.*

Corollary 1. " From this Demonstration it is manifest " that the Axis BO , of an Equilateral Cone, circumscrib'd " about a Sphere, is one and a half of the Diameter of the " Sphere EK , or as 3 to 2.

2. " That QT , the Base of the Cone DOF , is also " one and an half of both Bases of the Cylinder circum- " scrib'd about the same Sphere. For QT is to BPM , " as 3 to 1. Therefore QT is to BPM twice, as 3 is to 2.

3. " That the Superficies of the Cone DOF is one and " an half of the Superficies of the Equilateral Cylinder cir- " cumscrib'd about the same Sphere. For that * is double * *Per Co-*
to QT , while this is quadruple to BPM †. Therefore *rol. 1. p. 31.*
the conical Superficies will be to the cylindrical, as twice † 24 and
3 to four times 1; that is, as 6 to 4, or as 3 to 2. *26 of this.*

PROP. XII. Theorem.

THE whole Superficies of an Equilateral Cone *Fig. 28.*
circumscribed about a Sphere, is quadruple to the
whole Superficies of a Cone inscribed in the same
Sphere.

By the foregoing, the whole Superficies of the Equilateral
Cone DOF circumscrib'd, is to the Superficies of the Sphere,
as

as 9 to 4; and the Superficies of the Sphere is to the whole Superficies of the inscribed Cone SKT , as 16 to 9 (by 39. of this.) Therefore by Permutation of Equality of Proportion, the whole Superficies of the circumscribed Equilateral Cone is to the whole Superficies of the Equilateral inscribed, as 16 to 4, or as 4 to 1. *Q. E. D.*

PROP. XLII. Theorem.

Fig. 29. **A** Sphere hath that Proportion to BKC , an Equilateral Cone inscribed in it, which 32 hath to 9.

Let the Sphere and Cone be cut by a Plane passing through the common Axis $K O$, producing in the Sphere the greatest Circle $O F K I$, and in the Cone the Equilateral Triangle $B K C$. Then a Plane being drawn through the Center A perpendicular to $O K$, let the Hemisphere $F G K I$ be cut off, in which let the greatest Cone $F K I$ be understood to be inscribed. Now because (by *Cor. 5. p. 15. l. 4.*) the Side $B C$ of the Equilateral Triangle cuts off a fourth Part of the Axis $O K$, $P K$ will be to $A K$, as 3 to 2, that is, as 9 to 6. But the Base $Q T$ is to the Circle $O F K I$, that is, to the Base $N D$, as 3 to 4, that is, as 6 to 8, as appears from what was demonstrated, *Prop. 39.* Wherefore seeing the Proportion of the Cone $B K C$ to the Cone $F K I$, is (by *Schol. 2. p. 15. l. 12.*) compounded of the Proportion of the Altitude $P K$ to the Altitude $A K$ (that is, of the Proportion of 9 to 6) and of the Proportion of the Base $Q T$ to the Base $N D$ (that is, of the Proportion of 6 to 8) the Cone $B K C$ will be to the Cone $F K I$, as 9 to 8. Wherefore seeing (by 30 of this) the Sphere $C G$ is quadruple of the Cone $F K I$, the Equilateral Cone $B K C$ will be to Sphere $C G$, as 9 to 32. *Q. E. D.*

PROP. XLIII. Theorem.

Fig. 28. **A**^N Equilateral Cone circumscribed about a Sphere is eight-fold of an Equilateral Cone inscribed in the same Sphere.

Let SKT and DOF be the Equilateral Cones inscrib'd and circumscrib'd, and let OKB be the common Axis. Then let as well both the Cones as the Sphere be cut by a Plane.

Plane passing through the Axis; their Sections will be two Equilateral Triangles, and the greatest Circle B P M. About the Triangle D O F likewise let there be understood to be describ'd the Circle N D O F, and let the Axis O K B be produced unto N. Now because the Side D F of the Equilateral Triangle doth (by *Cor. 5. p. 15. l. 4.*) cut off N B. the fourth Part of the Axis O N, it is manifest that O N is double to B K. In like manner, because the Side S T of the other Equilateral Triangle cuts off B C, the fourth Part of the Axis B K, N O will be to B O, as B K is to C K; and by changing, as N O is to B K, so is B O to C K. But N O is double to B K. Therefore B O is likewise double to C K. Therefore because of the Similitude of the Triangles, D O F, S K T, D F and S T also, to wit, the Diameters of the conical Bases will (by 4. l. 6.) be in a double Proportion betwixt themselves. Wherefore seeing the Cones D O F, S K T, be like, and consequently (by 12. l. 12.) their Proportion is triplicate to the Proportion of the Diameters D F and S T; which is that of 2 to 1, the Cone D O F will be to the Cone S K T, as 8 to 1. Q. E. D.

B

PROP. XLIV. Theorem.

A Sphere hath the same Proportion both in respect of Solidity and Surface to the Equilateral Cone D O F circumscribed about it, which 4 hath to 9. *Fig. 28.*

The Sphere T P is (by 42 of this) to the Equilateral Cone S K T inscribed in it, as 3 is to 9. But (by the foregoing) S K T, the Equilateral Cone inscribed, is to D O F, the Equilateral Cone circumscribed, as 1 is to 8, that is, 3 is to 72. Therefore by Equality of Proportion, the Sphere T P is to D O F, the Equilateral Cone circumscribed, as 32 is to 72, that is, as 4 to 9. But in *Prop. 40.* we demonstrated that the Superficies of a Sphere is to the whole Superficies of an Equilateral Cone circumscribed, as 4 is to 9. Therefore a Sphere, both in Solidity and Superficies, is to an Equilateral Cone circumscribed about it, as 4 is to 9. Q. E. D.

That therefore which *Archimedes* was surpriz'd at in a Sphere and Cylinder encompassing it, we have also now demonstrated in a Sphere and an Equilateral Cone encompassing it, to wit, that there is the same rational Proportion of the Solidities betwixt themselves, which there is of the Surfaces.

For

For as he found that the Sphere is to the Cylinder, as well in Solidity as Superficies, as 2 to 3; so we have now taught, that the Sphere is, in respect both of Solidity and Surface, to an Equilateral Cone encompassing, as 4 to 9.

But from hence we shall, without much Labour, demonstrate, that the very Proportion, to wit, the Sesquialteral, which *Archimedes* shew'd to be betwixt the Sphere and Cylinder, is continued by the Equilateral Cone circumscribed both in the Solidity and Superficies; and so we shall put an End to the present Work.

P R O P. XLV. Theorem.

See the Figure prefixed to this Treatise.

AN Equilateral Cone circumscribed about a Sphere, and a right Cylinder in like manner circumscribed about the same Sphere, and the same Sphere, itself, continue the same Proportion; to wit, the sesquialteral, as well in respect of the Solidity as of the whole Superficies.

For by 32. of this Book, the right Cylinder G K encompassing the Sphere, is to the Sphere, as well in respect of Solidity, as of the whole Superficies, as 3 is to 2, or as 6 to 4. But by the foregoing, the Equilateral Cone B A D circumscribed about the Sphere, is to the Sphere in both the said respects, as 9 is to 4. Therefore the same Cone is to the Cylinder, both in respect of Solidity and Surface, as 9 is to 6. Wherefore these three Bodies, a Cone, Cylinder and Sphere, are betwixt themselves, as the Number 9, 6, 4, and consequently continue the sesquialteral Proportion. Q. E. D.

[P R O P. XLVI.]

THE same sesquialteral Proportion holds betwixt an Equilateral Cone and Cylinder circumscribed about the same Sphere, in respect of their whole Surfaces, their simple Surfaces, their Solidities, Altitudes and Bases.

This Proposition is manifest as to the whole Surfaces and Solidities from the foregoing; as to the simple Surfaces, from Cor. 3. Pr. 40. of this; as to their Altitudes and Bases, from Cor. 1. and 2. of the same 40th Proposition.]



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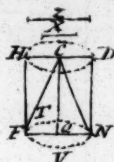
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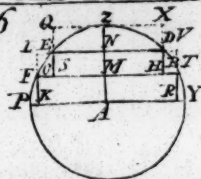
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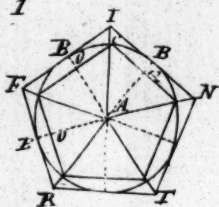
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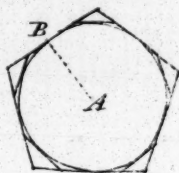
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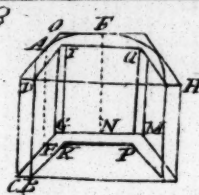


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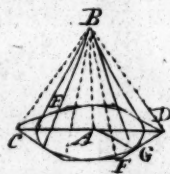
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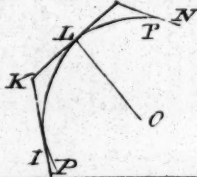
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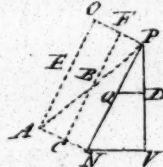
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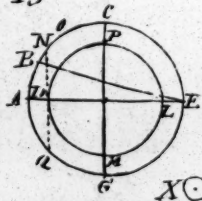
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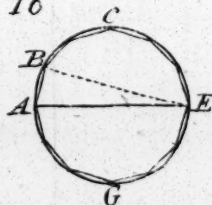
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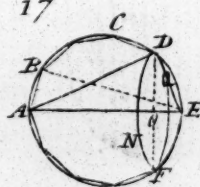
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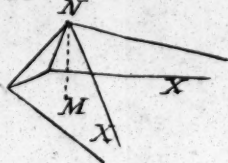
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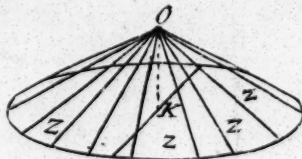
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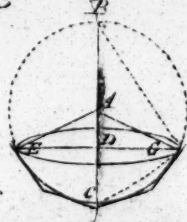
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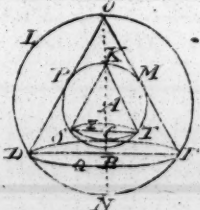
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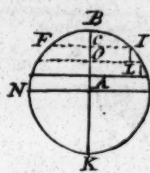
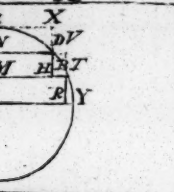
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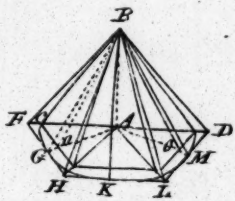


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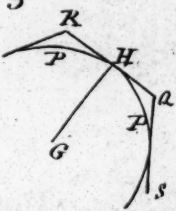


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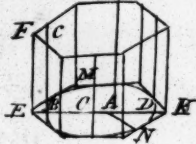
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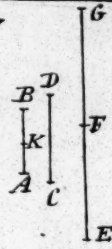
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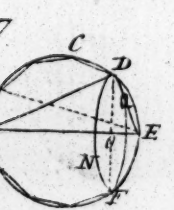
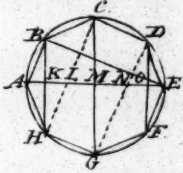
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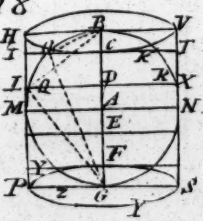
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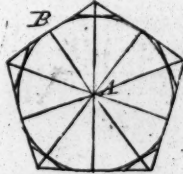
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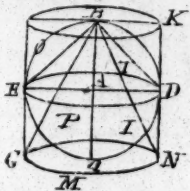
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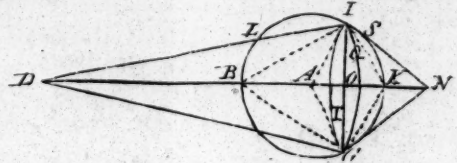
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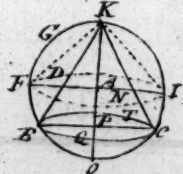
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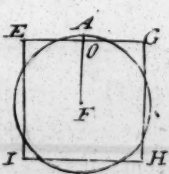
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PRACTICAL GEOMETRY.

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The FIRST contains an *Easy, Brief and Independent* Demonstration of certain Select and most Useful *Propositions* in *E U C L I D*.

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P A R T I.

WE often find that EUCLID, for the demonstrating one *important* Proposition, hath made use of a long Chain of others, which have no other End, but to demonstrate that *principal* One: If we can *all at once* demonstrate those *capital* Propositions, without such a *long Series* of preparatory Demonstrations; we shall doubtless retrench many useless Things, gain Time, and render this Appendix of some Service to the young *Student*.

There are two primary Propositions, of which *Des Cartes* writes in a Letter not yet printed, as hinted in *Page 65.* of Dr. PELL's *Algebra*; *In searching the Solution of Geometrical Questions, I always make use of Lines, Parallel and Perpendicular, as much as possible, (i. e. as many Lines as are useful) and I consider no other Theorems, but the 47th, L. I. and 4th, L. 6, of Euclid; and I am not afraid to suppose ma-*
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ny unknown Quantities, that I may reduce the proposed Question to such Terms, as to depend on no other Theorems, but these two.

Observe in how great Esteem the ingenious *Des Cartes* held these two *Theorems*; to the first of which, most of the preceding Propositions in *EUCLID* are preparative, and which we shall Essay to demonstrate *independently* of any of them.



EUCLID.



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P R O P. XLVII. Theorem.

*I*N every right-angled Triangle ($A B C$) the Square of the Side ($A C$) which is opposite to the Right-angle, is equal to the two Squares of the other Sides ($A B$ and $B C$) taken together.

In this are two Varieties; First, when the two Sides $A B$ and $B C$ are equal.

On the Side $A C$ erect the Square $A C D E$. on the Side $A B$ erect the Square $A B D G$, on the Side $B C$ erect the Square $B C E F$. *Fig. 1.*

Demonstration. 'Tis evident at first View of *Figure 1.* that one half of the Square of $A B$ or $B C$, is an exact quarter of the Square of $A C$, consequently four halves of the Squares $A B$ and $B C$, which are both their whole Squares, are equal to four quarters of the Square of $A C$, which is the whole Square of $A C$. *Q. E. D.*

Secondly, when the Sides $A B$ and $B C$, are unequal. *Fig. 2.* Make two equal Squares $B D E F$. $B D G H$ on the Line $B D$, equal to $A B$ and $B C$, the two less Sides of the given Triangle; in the first, on $A C$, erect the Square $A C I K$; in the other, on $A B$ and $B C$, erect the Squares $A D B L$ $= \square B C$ and $B C H M = \square A B$; and last of all, draw the Lines $A C$ and $B G$.

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Demon.

Demonstration. First, 'Tis evident from the Construction that the Squares B D E F and B D G H are equal.

Secondly, 'Tis obvious at first View, that the Square B D E F exceeds the Square of A C by four right-angled Triangles, whose Base is = A B and Perpendicular = B C, and consequently, by *Axiom 7.* equal to one another, and the given Triangle A B C.

Thirdly, 'Tis equally evident, B D G H exceeds the Squares of A B and B C, by four right-angled Triangles, whose Base is = A B and Perpendicular = B C, and consequently, by *Axiom 7.* equal to one another, and the given Triangle A B C, and also to the four right-angled Triangles aforesaid.

Fourthly, Wherefore, by *Axiom 3.* if from Equals, (the Squares B D E F and B D G H) be taken away Equals (the four right-angled Triangles in each Square) the Remains will be equal (the Square of A C in one Equal to the Squares of A B and B C in the other.) *Q. E. D.*

Or Algebraically, thus;

Let the biggest Square made of the Sum of the Sides = S S

$$\left. \begin{array}{l} \square A C = H H \\ \square A B = B B \\ \square B C = P P \end{array} \right\} \text{I say, } H H = B B + P P.$$

1. $S S = H H + 2 B C$, which $2 B C = 4$ Triangles.
2. $S S = B B + P P + 2 B C$, by $B + C \times B + C$.
3. $H H + 2 B C = B B + P P + 2 B C$.
4. $H H = B B + P P$. *Q. E. D.*

A like useful Theorem of signal Service in the whole Theory of Compound Motions, I shall subjoin.

In every Parallelogram the Sum of the Squares of the two Diagonals is equal to the Sum of the Squares of the four Sides.

To prove this by Trigonometry requires 21 Operations, by Analysis, or Algebra 15. which *M. de Lagny* has reduced to 7 Steps.

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4 Sides. $\left\{ \begin{array}{l} \square B C \\ \square A B \\ \square A C \\ \square D C \end{array} \right.$

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Howbeit the Reasonableness of this Theorem may appear *Fig. 3.* from this single Consideration, that all Triangles on the same or equal Base, and betwixt the same Parallels, are equal, compared with the 12th and 13th of the second Book.

In this are two Varieties; First, when the Parallelogram is right angled.

Then the Proposition is evident from the 47th, I. *Euclid*, just now demonstrated,

Secondly, But when the Angles of the Parallelograms are oblique, draw the prick'd Lines AG, BF, CH, DE, which being Perpendiculars betwixt the same Parallels, are all equal. Also GB and DH, which are also equal to AF and EC, being perpendicular betwixt the equal Parallels, let each of the first $4=y$ and each of the last $4=z$, also $AD=BC$ call x ; consequently $BE=x-z$.

Demonstration.

$$\left\{ \begin{array}{l} \square BC = x x. \\ \square AD = x x. \\ \square AB = y y + z z. \\ \square DC = y y + z z. \end{array} \right. \left\{ \begin{array}{l} \text{Diag.} \\ \square BD = y y + x x + z z - 2 x z. \\ \square AC = y y + x x + z z + 2 x z. \end{array} \right.$$

$$\text{Sum } 2 y y + 2 x x + 2 z z. = \text{Sum Dia. q. } 2 y y + 2 x x + 2 z z.$$

Corollary. " Hence 'tis plane, the Square of the longer
 " Diagonal exceeds the Sum of the Squares of the two
 " contiguous Sides, exactly by as much as the Square of
 " the shorter Diagonal wants thereof; that is to say, by
 " the double Rectangle xz , whose Length is the longest
 " Side, and Breadth equal to the Distance that the Perpen-
 " dicular from the opposite Angle falls from one End of it,
 " either within or without the Parallelogram.

E U C L I D. Book VI.

P R O P. IV. Theorem.

T*Triangles which are Equiangular to one another, are like or similar, that is, have their Sides also that are opposite to the equal Angles proportional.*

Fig. 4.

This I take to be the same with the first Definition of the same Book.

And that similar Triangles (ABC, ADE) have their corresponding Sides proportional, I shall illustrate from *Figure 4.*

Let the Triangle ABC be laid on the Triangle ADE, so will the Angle A, because equal in both, exactly coincide, and the Line AC fall on the Line AE, and AB on AD, by *Axiom 7.* and because the Angle C = Angle E, and the Angle B = Angle D, the Side BC will be parallel to DE, by *Defn. 1. Book 6.*

Then suppose AB a third of AD, make BF another third, and parallel to DE draw FG, then parallel to AD draw GH and CI; by which Parallels, the Sides AE and DE will also be divided into three equal Parts, by *Axiom 12.* Therefore it will hold as,

$$\begin{array}{rcl}
 AB : AD :: AC : AE & AC : AE :: AB : AD \\
 4 \quad 12 \quad 5 \quad 15 & 5 \quad 15 \quad 4 \quad 12 \\
 \text{that is,} & \text{that is,} \\
 AB : 3AB :: AC : 3AC & AC : 3AC :: AB : 3AB \\
 \hline
 AB : AD :: BC = DI : DE & CB = DI : DE :: AC : AE \\
 4 \quad 12 \quad 3 \quad 9 & 3 \quad 9 \quad 5 \quad 15 \\
 \text{that is,} & \text{that is} \\
 AB : 3AB :: DI : 3DI & DI : 3DI :: AC : 3AC
 \end{array}$$

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To Trisect an Arch of a Circle B C.

This has been by sundry Antient and Modern Geometicians accounted impossible to be done by the *Euclidean* Geometry, which makes use of only a Circle and strait Lines; howbeit, we will attempt it, and afterwards a Demonstration thereof.

Quarter the Circle, and extend the Diameter B G to P; *Fig. 5.* parallel to it, drawn C X and C R parallel to E X; then take the Diameter B G in the Compasses and move the Ruler on the Point C, till F P be exactly equal to B G, then draw C F O P, so will G O be a third of B C.

Demonstration.

Through the Center draw O E Z. Now because E O is Radius and E P the Diameter, and the Angle F E P is right; therefore the Lines F O, E O, P O, are all equal, and also the external Angle F O E = 2 P E O, the two internal opposite Angles, or 2 Z E B, which is Vertical, and consequently equal to P E O. But F O E, that is, C O Z; being in the Periphery, is measured by half the Arch C Z; wherefore B Z, which is the Measure of half the Angle C O Z, is a fourth of the whole Arch C Z. and consequently a third of B C; and therefore G O = B Z is also a third of B C.
Q. E. D.

Note, If the Arch to be trisected be greater than a Quadrant, then trisect its Complement to 180, and the third of this Complement, taken from 60 Degrees always leaves the third of the Arch required.

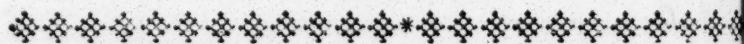
Left the foregoing Demonstration should appear too concise to some, I will attempt it after another Manner, from the following *Lemma*.

“That the Measure of the Angle C P B, at a Point without the Circle is equal to half the Difference betwixt the intercepted Arches C B and O G, or to the Difference betwixt half their Sum C O Z, and the less Arch O G, which is the same.

For $BZ = OG$, because Vertical Angles, by 15. *l. 1.* then is CZ the Sum of the intercepted Arches CB and OG , and COZ , the Angle at the Circumference, the Measure of half the said Sum, by 20. *l. 3.* which COZ , being external, is equal to the two internal and opposite Angles OEP and OPB , by 32. *l. 1.* but the Measure of OEP is OG , the less Arch: Now, if $COZ = OG$, and the Angle $OPB = CPB$, I say, CPB must be equal to the Difference betwixt COZ and OG . *Q. E. D.*

Corollary. "In the said Triangle EOP , If the Angles "at E and P be both equal, then will OG be a third of " BC ; because OG will be half of COZ , or a fourth of " the whole Arch CZ , and consequently a third of BC " $= \frac{1}{3} BC$, for from the whole CZ , take away BZ , " remains $BC = \frac{2}{3} CZ$. It remains only to prove the Angles " E and P are equal.

Because FEP is a right Angle by Construction the Center of the Semi-circle FEP will fall in O , which bisects FP , the Diameter, which is double EO , the Radius; consequently all three, FO , EO , PO , are equal Radii of the Semi-circle FEP aforesaid; and because EO and PO are equal, the Angles at the Base E and P , by 5. *l. 1.* are equal.



A Synopsis of the most useful and famous Propositions in EUCLID's first Six Books.

BOOK I. has *Propositions* 32, 35, 37, 41, 44, 45, 47.
 II. ————— 4, 5, 6, 12, 13, 14.
 III. ————— 16, 20, 21, 22, 31, 32, 35, 36.
 IV. ————— 10, 11, 12, 13, 14, 15, 16.
 V. ————— { 15, 16, 17, 18, or the foregoing Compend. thereof.
 VI. All the first 6, 8, 13, 14, 16, 19, 31.

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Extraordinary Propositions not to be met with in EUCLID, whose Demonstrations are omitted on purpose to Exercise the Genius of young Mathematicians.

1. **I**N every Triangle the Rectangle of any two Sides is equal to the Rectangle of the Perpendicular from the said Angle, and the Diameter of the circumscribing Circle, *i. e.* as Perpendicular : one Side :: other Side : Diameter.

2. The Area of any Triangle about a Circle is equal to the Rectangle of the Semi-diameter, and half the Sum of the Sides

3. An Hexagon inscribed is a Mean betwixt a Trigon inscribed, and a Trigon circumscribed, *Et sic de paribus,*

4. In any Triangle the Difference of the Squares of two Sides is equal to a Rectangle of the Base, and that Segment of the Base, which parted, in the Middle of the other Part the Perpendicular falls.

5. The Square of the Mean, and the Square of half the Difference of the Extremes together, is equal to the Square of the half Sum of the Extremes.

6. If in a Circle two Lines be inscribed, intersecting each other, the Rectangles of the Segments of each Line are equal; and the Angle at the Point of Intersection is measured by half the Sum of its intercepted Arches.

7. If to a Circle two right Lines be adscribed, from a Point without the Rectangles of each Line, from the said Point to the Convex and Concave are equal, and the Angle at the Point is measured by half the Difference of the intercepted Arches.

8. If in a Circle three right Lines shall be inscribed, one of them cutting the other two, then the Rectangles of the Segments of each Line so cut, are directly proportional to the Rectangles of the respective Segments of the said cutting Line.

9. If a plain Triangle be inscribed in a Circle, the Angles are one half of what their opposite Sides do subtend; and if it hath one right Angle, the longest Side of that Triangle shall be the Diameter of the Circle.

*Easy Equations arising by comparing two unequal
fitions of the Second Book of EUCLID are invented.*

The Reason.	Numb. Equat.	In Species.	In Numbers.
	1	$z = m + n$	$10 = 7 + 3$
	2	$x = m - n$	$4 = 7 - 3$
$1 - n$	3	$m = z - n$	$7 = 10 - 3$
$1 - m$	4	$n = z - m$	$3 = 10 - 7$
$2 + n$	5	$m = x + n$	$7 = 4 + 3$
$2 - x$	6	$n = m - x$	$3 = 7 - 4$
$1 + 2$	7	$2m = z + x$	$14 = 10 + 4$
$1 - 2$	8	$2n = z - x$	$6 = 10 - 4$
$7 \frac{..}{2}$	9	$m = \frac{z + x}{2}$	$7 = \frac{10 + 4}{2}$
$8 \frac{..}{2}$	10	$n = \frac{z - x}{2}$	$3 = \frac{10 - 4}{2}$
$7 - x$	11	$z = 2m - x$	$10 = 14 - 4$
$7 - z$	12	$x = 2m - z$	$4 = 14 - 10$
$8 + x$	13	$z = 2n + x$	$10 = 6 + 4$
$15 - 2n$	14	$x = z - 2n$	$4 = 10 - 6$
$2 \times x$	15	$xx = xm - xn$	$16 = 28 - 12$
3×4	16	$mn = zz + mn - zm - zn$	$21 = 100 + 21 - 70 - 30$
5×5	17	$mm = xx + 2xn + nn$	$49 = 16 + 24 + 9$

Quantities
Let m

The Reason.

$1 \times b$
 $1 \times z$
 $1 \times m$
 $1 \times n$
 1×1
 2×2
 7×8

24

25 +

1X
 $27 + mn$
 3×3
 $29 + 2z$
 $24 + xx$
 $22 + 23$

32
2

Quantities and Numbers, whereby most of the Propo-
Let m=7 n=3 z=10 and x=4.

The Reason.	Numb. Equat.	In Species.	Propositions of the Second Book
$1 \times b$	18	$zb = bm + bn$	1
$1 \times z$	19	$zz = zm + zn$	2
$1 \times m$	20	$zm = mm + mn$	3
$1 \times n$	21	$zn = nn + mn$	
1×1	22	$zz = mm + 2mn + nn$	4
2×2	23	$xx = mm - 2mn + nn$	
7×8	24	$4mn = zz - xx$	
..		$zz - xx$	
24 $\frac{..}{4}$	25	$mn = \frac{4 \quad 4}{4 \quad 4}$	
xx		$zz \quad xx$	
25 $\frac{xx}{4}$	26	$\frac{..}{4} = mn + \frac{xx}{4}$	5
1×2	27	$zx = mm - nn$	
$27 + nn$	28	$zx + nn = mm$	6
3×3	29	$mm = zz - 2zn + nn$	
$29 + 2zn$	30	$mm + 2zn = zz + nn$	7
$24 + xx$	31	$4mn + xx = zz$	8
$22 + 23$	32	$zz + xx = 2mm + 2nn$	10
		$zz + xx$	
32 $\frac{..}{2}$	33	$\frac{2}{2} = mm + nn$	9
		2	
		And so infinitely.	

CCXVZD (X) CCXVZD (X) CCXVZD (X) CCXVZD

PRACTICAL GEOMETRY.

PART II.

AS to the Demonstration of these *Practical Problems*, I have purposely omitted them, that the young Student may exercise his own Genius, in making Application of, and recollecting what he has already learned from the foregoing Books of *EUCLID*; to awaken the Mind, to whet the Appetite of our Mathematical Student, and to amuse and improve him in easy and practical Problems of Geometry, is the main Design of this Appendix.

And lest we go over what is already done, I think it not amiss to give a *Synopsis* of all the *Practical Problems* both in the *Propositions* and *Corollaries* of the foregoing Book, referring to the Book and Proposition where they may be found

In the First Book.

PROPOSITION 1. On a given Line to make an Equilateral Triangle.

To measure inaccessible Distances, see 4. and 26. also L. 6. Prop. 8.

2. To draw a Line equal to a given one.
3. From a greater to cut off a less.
4. To play at Billiards.
6. To measure accessible Altitudes, see 33. also L. 6. 4.
9. To bisect an Angle.
10. To bisect a Line.
11. To erect a Perpendicular, and L. 3. 31.
12. To let fall a Perpendicular.
15. That Rays of Light reflected, take the shortest Course.
19. A Globe can rest no where, but in the Point it touches the Earth.
22. To make a Triangle of three Lines given.

22. To

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22. To make an Angle equal to a given one.
To measure a given Angle.
To lay down an Angle of any Number of Degrees.
27. To measure the Compass of the Earth, see *L. 2. 6.*
31. To draw Parallels.
32. To determine the Parallax of the Stars.
To find the Number of right Angles contained in the Angles of any right-lined Figure.
33. The Demonstration of Compound Motions.
34. To divide the Area of a Parallelogram.
36. Figures of equal compass may have different Area's.
38. To divide the Area of a Triangle; that Bodies move equal Area's in equal Times.
40. And are urged by a Centripetal Force.
41. To find the Area of a Triangle, *L. 2. 13.*
42. To make a Parallelogram with an Angle equal to a given one, and equal to a Triangle given.
44. On a given Line, to make a Parallelogram equal to a Triangle given, and to have an Angle equal to one given.
To demonstrate Geometrical Division.
45. On a given Line, and with a given Angle, to make a Parallelogram equal to any strait-lined Figure.
To find how much one strait-lined Figure exceeds another.
46. To make a Square on a given Line.
47. To add any Number of Squares.
To take a less Square out of a greater; any two Sides of a right-angled Triangle, to find the third.
The Origine of the Table of Sines Tangents and Secants. See *L. 3. 3. L. 4, &c.*

In the Second Book.

11. To cut a Line in extreme and mean Proportion, *L. 6. 30.*
14. To find a Square equal to any right-lined Figure.

In the Third Book.

1. To find the Center of a Circle.
12. To find the Point where two Circles touch each other.
16. To demonstrate the infinite Divisibility of a strait Line. See *L. 1. 47.*
17. To draw a Tangent to a Circle.

20. To

20. To demonstrate, the Sides of Triangles are in such Proportion as the Sines of their opposite Angles.
To measure the Distance of the Sun or Moon.
21. How the same Line, at different Distances, may appear of the same Length.
22. About what Quadrangle a Circle can or cannot be described, *L. 6. 4.*
25. To perfect an Arch into a Circle.
30. To bisect a given Arch.
31. To prove whether a Square be true.
33. On a Line to form a Segment capable of any given Angle.
34. From a Circle, to cut a Segment containing any given Angle.
35. Certain Geographical Paradoxes solved; to draw a Circle through any two Points in another given Circle, which shall diametrically cut it.

In the Fourth Book.

1. To inscribe a Line in a Circle.
2. To inscribe a Triangle in a Circle.
3. To circumscribe a Triangle about a Circle.
4. To inscribe a Circle in a Triangle.
5. To circumscribe a Circle about a Triangle.
- 6, 7. To inscribe and circumscribe a Square in or about a Circle,
- 8, 9. To inscribe and circumscribe a Circle about a Square.
10. To make an Isosceles-Triangle, whose Angle at the Base is double the Angle at the Vertex.
11. To inscribe a regular Pentagon, or any other Polygon in a Circle: On a given Line to describe a Pentagon.
12. To circumscribe a regular Pentagon, or any Polygon.
- 13, 14. To inscribe in, or circumscribe a Circle about a regular Polygon.
15. On a Line given to describe an Hexagon, or to inscribe it, or an Equilateral Triangle in a Circle.
16. To inscribe a regular Quindecagon, or innumerable regular Figures.

On a given Line to describe any regular Figure,
What number of regular Figures will fill a Space, *i. e.*
whose Angles about one Point, just make 360 Degrees.

In

In the Sixth Book.

1. To divide a Trapezium.
6. To make Similar Triangles.
9. To divide a Line in a given Proportion.
10. To divide a Line as another is, or into any Number of Parts equally.
11. To find a third Proportional to two given Lines, also the Sum of infinite Proportionals.
12. To find a fourth Proportional to three Lines given.
13. To find a mean Proportional, or two Means betwixt two given Lines divers Ways, *L. 12. 18.*
14. To demonstrate the Inverse Rule of Proportion.
16. To demonstrate the direct Rule of Proportion,
18. On a right Line to describe a Polygon like and alike situate, to a given one. Hence the drawing and reducing Maps. See 20, 21.
19. Similar Figures are in a duplicate Ratio of like Sides. See 20. 23. *L. 12. 2.*
25. To make a Polygon equal to a given one, and like to another given one.
31. To add or subtract right-lined Figures, and to square the Lunets of *Hippocrates*.

In the Eleventh Book.

11. To draw a Perpendicular to a Plane.
12. To erect a Perpendicular on a Plane.
21. A Demonstration of the Five regular Bodies.
33. Like Bodies are to each other, as the Cubes of their homologous Sides, *L. 12. 9.*

In the Twelfth Book.

18. To encrease or diminish Solids.

By this Synopsis, and the following Appendix, it will appear what a large *Apparatus* towards a System of *Prædictæ Geometry* is already Extant in our Language for want of which there has been, these many Years past, no small Complaint: Especially if to this be added *Le Clerc's* Geometry, *Hawney's* Complete Measurer, and *Langley's* Practical Geometry; to the first of which, I am not a little in debt for several of the following Problems.

Problem

Problem I.

TO divide a right Line (A B) into any Number of equal Parts, suppose 8.

For this Purpose have in readiness a Number of equidistant Parallels drawn and numbered from 0 to 12 or 20, with the distance of A B given Set one Foot of the Compasses on the Parallel marked 0.0. and turn the other till it touch the eighth Parallel; unto which two Feet apply a Ruler, and draw the Line A B, which the Parallels will equally divide into eight Parts. *W. W. D.*

This is so plain and easy, as not to need any Figure.

Problem II.

TO find the Area of a plain Triangle, having the three Sides, without a Perpendicular.

Add the three Sides, and take half that Sum; then subtract each Side severally from that half Sum: Multiply that half Sum and the three Differences continually, and out of the last Product extract the square Root, which is the Area of the given Triangle.

Or,

Having the Diameter of the inscribed Circle, multiply it by a fourth of the Sum of the three given Sides for the Area sought.

Problem III.

HAVING the Area of any Triangle, to find the Diameter of the inscribed Circle.

Divide the said Area by a fourth of the Sum of the three Sides; the Quotient is the Diameter of the inscribed Circle.

Problem IV.

TO reduce any Figure into a Square; or to make a Square equal to a Parallelogram, Triangle, Trapezium, Polygon or Circle, &c.

By the Rules of Mensuration, cast up their Area's, the square Root whereof is the Side of the Square sought.

Or

Confide
their Area
Geometric
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Squares, which Squares add by 47. 1. or 14. 2. Euclid.

Or Geometrically thus :

Consider what two Things multiplied together produce their Area : Between those two Numbers, or Sides, find a Geometrical Mean by *Lib. 6. 13 Euclid.* This Mean is the Side of a Square equal. *W. W. D.*

<i>N.B.</i> The two Things, which multiplied, produce their Area in a	Parallelogram rectangled,	{ are { The Length and Perpendicular Breadth.
	Rhombus, — — — —	
	Rhomboides, — — — —	
	Triangle, — — — —	
	Trapezium, — — — —	
{ are {	Circle, — — — —	{ Base and half Perpendicular.
	Polygon, — — — —	
	Sector, — — — —	
	Segment, — — — —	
<i>N.B.</i> If it be an irregular Figure, reduce it into Triangles, or Trapezia's, by drawing Lines within from Corner to Corner; then reduce the said Triangles, &c. into Squares, which Squares add by 47. 1. or 14. 2. <i>Euclid.</i>	Half Perimeter and the Perpendicular from the Center.	{ Diagonal and half of both Perpendiculars.
	Half the Arch and half the Diameter.	
	Chord, and two Thirds of the V. Sine.	

Probiem

Problem V.

TO find a strait Line (E I) nearly equal to the Arch of a Quadrant (B C.)

Fig. 6.

Bisect the Quadrant B D in F, and draw B F; to B F erect the Perpendicular A B on B; then set A C from C to I, so will E I be nearly equal to the Arch of the Quadrant B D or B C.

Or,

Bisect B D in F, and B C in G, then with the Distance K F, on the Center K, draw the Arch H F G I, so will the Line H I be nearly equal to the Semi-circumference, and E I to the Quadrant, as before.

N. B. *Archimedes* demonstrated, that the Circumference is to the Diameter less than as 22 : 7. and greater than $21\frac{7}{11}$ to 7; within which strict Limits, many Years ago, it was found in whole Numbers to be as 9 : 10 :: Chord : Arch of a Quadrant.

Suppose then that the Diameter = 7, its half 3.5, the Square of which is 12.25; which doubled, is 24.5, whose Square Root is 4.949, the Chord of the Quadrant; then say, as 9 : 10 :: 4×4.949 : 21.995, the Circumference, which is greater than $21\frac{7}{11}$, or 21.985. Again, 4.95 is something greater than the Chord, which $\times 40$, and divide by 9, as above, gives 22; so that by this Proportion, as 9 : 10. a like Answer with *Archimedes* is also discovered, i. e. less than as 22 : 7, and greater than as $21\frac{7}{11}$ to 7.

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Concerning Squaring the Circle, I offer some few Easy and Practical Observations following.

1. **A** Circle is equal to a right-angled Triangle, whose Base is the Circumference of the Circle and its Perpendicular the Radius of it.
2. Every Polygon circumscribed is greater, and every Polygon inscribed is less than the Circle.
3. The Compass of a Polygon circumscribed is greater, and inscribed is less than the Circumference of a Circle.
4. This right-angled Triangle aforesaid, will be less than any Polygon circumscribed, and greater than any inscribed: Because the Circumference of this Circle (which is the Base of this Triangle) is greater than the Compass of any inscribed, and less than the Compass of any circumscribed Polygon; therefore it will be equal to the Circle.

Because every imaginable inscribed Figure, which is less than the Circle, is also less than the Triangle; and every circumscribed Figure, greater than the Circle, is also greater than the said Triangle likewise; therefore the said Triangle is equal to the Circle: But actually to find a right Line exactly equal to the Circumference, is not yet discover'd Geometrically.

Howbeit, I shall offer an easy Approximation to the young Student, whom I desire to account the Circle a Polygon of 1000, 10000 or 43200 Sides, *i. e.* the half Minutes in 360, so small a Part of a Circle will insensibly approach to, or become very nearly a strait Line; nay, the Product of the Sine and Tangent of one minute multiplied by the said 43200, will agree in the last 7 Figures, *i. e.* 6283185, whereby we obtain the Circumference of the Circle, whose Radius is 1000000, to the like Number of Figures exactly, and may to as many more Places, if we add the two Products together, and take the half for the Circle's Circumference, which will be less than the *Tangent-Product*, or circumscribed Polygon, and greater than the *Sine-Product*, or inscribed Polygon.

S

For,

For, according to the preceding third Proposition out of *Archimedes*, the Circuits or Polygons circumscribed about and inscribed in a Circle, do at last end in the Circumference of the Circle; in like manner the Polygons themselves do at last end in a Circle.

To this I subjoin certain Practical Remarks on Regular Polygons, their Angles and Sides.

Regular Polygons may be delineated several Ways; First, By the Angle at the Center; Secondly, By the Angle at the Figure; Thirdly, By the Angle of the Triangle at the Base.

To find Each.

Divide 360 by the Number of Sides, the Quotient is the Angle at the Center, whose complement to 180 is the Angle at the Figure, and half the said Complement is the Angle at the Triangle: By this one may make a Table of Angles for as many Polygons as he pleases.

Polygons of odd Number of Sides are inscribed in Circles by the help of *Isoceles* Triangles, whose Angles at the Base are Multiples of that at the Top.

If the Angle at the Base be $\left\{ \begin{array}{l} \text{Double} \\ \text{Triple,} \\ \text{Quadruple,} \\ \text{Quintuple,} \end{array} \right\}$ the Angle at the Vertex, the Base will be $\left\{ \begin{array}{l} \text{Pentagon.} \\ \text{Heptagon.} \\ \text{Enneagon.} \\ \text{Endecagon.} \end{array} \right\}$

The Number of Degrees of the Angle at the Vertex, is found by dividing 180 by the Number of Sides in the Polygon to be inscribed, the Quotient gives the said Angle; which doubled, tripled, &c. will give the Angle at the Base.

Seeing Polygons inscribed are the Chords of the Angles at the Center, which Chords are always double the Sine of half the Angle at the Center. Therefore in the Table of Natural Sines, hunt out the Sine of half the Angle at the Center, which doubled, is the exact Side of the Polygon, in such Parts as the Radius contains 10000000, &c.

A Table

Num	Sid
3	4
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12

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A Table

A Table for the inscribing and describing Polygons.

Number of Sides.	Quantity of the Side	Angles at the Center.	Angles at the Figure.	Angles at the Triangle.
3	17320508	120	60	30
4	14142135	90	90	45
5	11755705	72	108	54
6	10000000	60	120	60
7	8677674	$51\frac{3}{7}$	$128\frac{4}{7}$	$64\frac{2}{7}$
8	7653668	45	135	$67\frac{1}{2}$
9	6840402	40	140	70
10	6180339	36	144	72
11	5634651	$32\frac{8}{11}$	$147\frac{3}{11}$	$73\frac{7}{11}$
12	5176380	30	150	75

The Use.

To inscribe a Heptagon in a Circle, whose Radius is 500;
say as *Tabular Radius* 1000: *Tabular Heptagon* 867:: so is
the Given Radius: Side Heptagon sought 433.

To describe a Heptagon on a given Line, find the Radius
proper; say as the *Tabular Heptagon* 867: *Tabular Radius*
1000:: so is the Given Side 433: to the proper Radius
sought 500.

Problem VI.

TO make a Circle (ADG) nearly equal to a
given Square (AECF.)

Draw the Diagonal AC, which divide into 10 equal
parts; 8 of them are nearly equal to the Diameter of the
Circle sought: Wherefore in the middle of the Diagonal
AC, and with the Radius of 4 of those equal Parts draw
the Circle ADG. W. W. D. Fig. 8.

Or thus more briefly;

Bisect EC in D, and draw AD, the Diameter of the
Circle sought.

Problem VII.

TO make an Ifoceles Triangle (BCE) equal to
a given Square (ABCD.)

S 2

Extend

Fig. 8.

Extend the Side CD to E , making $DE = CD$, and draw BE , so will BCE be equal to $ABCD$. *W. W. D.*

To make an Equilateral Triangle equal to a Square, some advise to make the Side of the Triangle one and a half of the Side of the Square; but it is somewhat too little

Problem VIII.

Within a given Triangle (KLM) to inscribe a Square ($NOPQ$.)

Fig. 9.

On M erect the Perpendicular MH , equal to LM ; let fall the Perpendicular KG , then draw GH , cutting KM in N ; through N draw NO , parallel to the Base LM , and NQ and OP parallel to KG , so is $NOPQ$ the greatest inscribed Square. *W. W. D.*

Problem IX.

Within the Square ($RSTV$) to inscribe an Equilateral Triangle RZY .)

Fig. 10.

Draw the two Diagonals ST and RV ; on V , as a Center, with the Extent VX , draw the Arch ZXY ; then draw the Lines RZ , ZY , RY , which form the Triangle RZY . *W. W. D.*

Problem X.

ABOUT an Equilateral Triangle (ABC) to describe a square ($AFDG$.)

Fig. 11.

Bisect BC in H , and draw AHD ; make HD equal to HC , and bisect AD in E ; through E draw FEG at right Angles, making FE and EG each $= ED$. then draw the Lines AF , FD , DG and AG , forming the Square *W. W. D.*

Problem XI.

ABOUT a Square ($LMNO$) to draw a Triangle (PQR) whose Angles shall be equal to the several Angles of a Triangle given (STV .)

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Extend $N O$ to $P Q$, then make the Angle $R L M = T$ Fig. 12.
and $R M L = V$, then draw the Sides $R L P$ and $R M Q$,
so is $R P Q$ the Triangle. *W. W. D.*

Problem XII.

*TO inscribe an Equilateral Triangle ($A F G$) in a
given Pentagon ($A B C D E$.)*

Find its Center H , and draw the Circle about it; then Fig. 13.
inscribe a Triangle in that Circle, which will also be the
Triangle in the Pentagon. *W. W. D.*

Or,

With the Radius $A H$, and Center A , draw the Arch
 $H M$, which bisect in K , and draw $A K F$, the Side of the
Triangle sought.

Problem XIII.

*TO inscribe the Square ($M N O P$) in the Penta-
gon ($A B C D E$.)*

Extend the Perpendicular $A Q$ and $B C$ till they cut each Fig. 14.
other in F ; make the Perpendicular $F I$ and $A H$ equal to
half $A F$, then draw $B H$ and $B I$, cutting $A Q$ in L and
 K , so will $L K$ be the Side of the Square sought.

Problem XIV.

*OF four given Lines ($A B=6$, $B C=9$, $C D=8$,
 $A D=18$;) the greatest being less than the Sum
of the rest, to make a Quadrangle which may be in-
scribed in a Circle.*

In every Quadrangle inscribed in a Circle, the Sum of Fig. 15.
the Rectangles made of the Sides, containing opposite An-
gles, have the same Proportion to each other as the Dia-
gonals, *i. e.* as,

$$\begin{array}{ccccccc} 72 & 108 & 54 & 144 & & & \\ DCB + DAB & \text{to} & ABC + CDA, & \text{so is} & CA & \text{to} & DB \\ 180 & & 198 & & 13+ & 15+ & \end{array}$$

S 3

Also,

Also,

$$\begin{array}{ccccccc} 144 & 54 & 72 & 108 & 162 & 48 & \\ \text{ADC} + \text{ABC} : \text{BCD} + \text{DAB} :: \text{AD} \times \text{BC} + \text{AB} \times \text{CD} : \text{AC} \square & & & & & & \\ 198 & & 180 & & 210 & & 190\frac{1}{2} \end{array}$$

The Root is 13+

Again,

$$\begin{array}{ccccccc} 72 & 108 & 144 & 54 & 162 & 48 & \\ \text{BCD} + \text{DAB} : \text{ADC} + \text{AEC} :: \text{AD} \times \text{BC} + \text{AB} \times \text{CD} : \text{BD} \square & & & & & & \\ 180 & & 198 & & 210 & & 231 \end{array}$$

The Root is 15+

From the last of these Proportions find the Diagonal BD; with which, and the other two Sides, either BC and CD, or AB and AD, form a Triangle, and about it draw the Circle ABCD, and in it insert the other two Sides of the Quadrangle. *W.W.D.*

Problem XV.

TO inscribe an Heptagon, Nonagon, or Undecagon, &c.

Fig. 16. Having obtained the Side of a Polygon, next bigger and next less, in the same Circle, extend the Diameter AB to D, and from C extend the said Sides of the Polygon next above or under to D and E, which Distance bisect in F, and draw CF, the Side of the Polygon sought lies on that Line CF, between C and the Circumference.

The Side of a Septagon or Heptagon, is nearly equal to the Perpendicular of an Equilateral Triangle, whose Side is Radius, or an Hexagon; so the Side of a Nonagon is nearly the Perpendicular of an Equilateral Triangle, whose Side is an Octagon.

Problem XVI.

TO describe a regular Octagon on a given Side (AB.)

Fig. 17. On the middle of AB, erect the Perpendicular CE; on the Point C and Distance AC, describe the Semi-circle ADB; on

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on the Point D, and with the Distance D A, draw A E B; so is E the Center, and A E the Radius of that Circle, which contains the Octagon, whose Side is A B. *W. W. D.*

Problem XVII.

TO describe a regular Nonagon on a given Line (A B.)

Erect the Perpendicular F C on the middle of A B; on Fig. 18. B, with the Distance A B, draw the Arch A D, which bisect in E; on the Point D, with D E, draw the Arch E F; so is F the Center, and A F the Radius of that Circle which contains the Nonagon, whose Side is A B. *W. W. D.*

Problem XVIII.

TO inscribe a regular Nonagon in a given Circle.

On B, with the Radius A B, draw D A C and D C, Fig. 19. which extend to F; make E F=A B; on E and F draw E G and F G, then draw A G, which cuts the Circle in H; so is D H the Side of the Nonagon sought. *W. W. D.*

H B is the Side of a regular Polygon of 18 Sides, and $\frac{1}{3}$ of the Radius is the Side of a regular Nonagon.

Problem XIX.

TO inscribe a regular Undecagon in a Circle.

Divide A B in two at C; on A and C, with the Distance Fig. 20. A C; draw the Arch C D I and A D; on the Center I and Distance I D, draw the Arch D O; so is C O the Side of the Undecagon sought.

Some say $\frac{2}{3}$ of the Diameter is the Side of an Undecagon inscribed, or $\frac{1}{3}$ Diameter more, $\frac{1}{8}$ of the said $\frac{2}{3}$.

Or,

Quarter a Circle and inscribe an Equilateral Triangle, that Part of the Side which lies betwixt the Diameter and the Angle of the Base is the Side of a regular Undecagon.

Problem XX.

TO describe a regular Dodecagon on a given Side (*A B*.)

Fig. 21.

On the Middle of *A B*, erect the Perpendicular *C D*; on *A* and *B*, with the Distance *A B*, draw the Arches *A E* and *B E*; on *E*, with the Distance *A E*, draw the Arch *A D*; so is *D* the Center of the Dodecagon sought: For *A E B* is the Angle of the Hexagon, and *A D B* is its half, the Angle of a Dodecagon.

Problem XXI.

On a given Line (*A B*) to describe any Polygon from 6 to 12 Sides,

Fig. 22.

Bisect *A B* in *O*; erect on *O*, the Perpendicular *O I*; on *B*; with the Distance *A B*, describe the Arch *A C*, which divide into 6 equal Parts from *C*, and from the Distance of each of those equal Parts draw the Arches *D M*, *E N*, *F P*, *G Q*, *H R* and *A I*; so is

$$\left\{ \begin{array}{c} C \\ D \\ E \\ F \\ G \\ H \\ I \end{array} \right\} \text{Center of a } \left\{ \begin{array}{c} \text{Hexagon,} \\ \text{Heptagon,} \\ \text{Octagon,} \\ \text{Nonagon,} \\ \text{Decagon,} \\ \text{Endecagon,} \\ \text{Dodecagon,} \end{array} \right\} \text{ and } \left\{ \begin{array}{c} C A \\ D A \\ E A \\ F A \\ G A \\ H A \\ I A \end{array} \right\} \text{ the Radius.}$$

Problem XXII.

TO cut two given Lines (*A B* and *A C*) into four, so that all four shall be continually proportional, (*C F : D F :: E D : B E*.)

Fig. 23.

Set *A B* and *A C* at right Angles, and draw *B C*; bisect *A B*, and draw the Semicircle *B D O* through *D*; draw *D F* parallel to *B O*, and *D E* parallel to *C O*; so will it be,

$$\begin{array}{l} B E : E D :: E D : D F \\ E D : D F :: D F : F C \end{array}$$

Problem

Problem XXIII.

FROM a strait Line given (*I*) to cut off a Part (*GD*) which shall be a Mean betwixt the other Part and another Line given (*H*.)

Make the Line $C D =$ Lines *H* and *I* given, on *E*, Fig. 24 where the two Lines meet, erect the Perpendicular *EF*, and draw the Semi-circle on the Diameter *CD*. Bisection $C E = H$ in *B*; on *B*, with the Radius *BF*, draw the Arch *FG*; so will $D G ; G E ; E C ::$.

Problem XXIV.

GIVEN the Sum of the Extremes (*AB*) and the Mean (*BD*) to find the Extremes (*AF* and *FB*) severally.

On *AB* erect a Semi-circle, and on *B* erect a Perpendicular Fig. 25. lar equal to *BD*; through *D*, parallel to *AB*, draw *ED* and *EF* parallel to *DB*; so is *AF* one and *FD* the other Extreme. *Q. E. I.*

Problem XXV.

GIVEN the Difference of the Extremes (*AB*) and the Mean (*BC*) to find the Extremes (*BE* and *BF*) severally.

On *AB*, the Difference, erect the Perpendicular $B C =$ Fig. 26. Mean. Bisection *AB* in *D*; on *D*, with the Distance *CD*, draw the Semi-circle, whose Diameter *FE* contains the two Extremes sought, *i. e.* *BE* and *BF*.

Problem XXVI.

THE Excess of the Diagonal above the Side of a Square being given (*AB*) to find the Side (*AD*).

Arith-

Arithmetically thus:

Fig. 27. To the Excess given AB , and the square Root of double the Square of the Excess for the Side AD sought,

Geometrically:

On B erect the Perpendicular $BC = AB$, and draw ACD ; on C , with the Distance CB , draw the Arch BD ; so will AD be the Side of the Square, and AE the Diagonal, which exceeds AD by AB .

Problem XXVII.

GIVEN the Sum of the Side and Diagonal (AB) to find them separately (GA and GB .)

Fig. 28. On AB draw the Semi-circle ACB ; erect CD perpendicular. Bisect AD in E ; on E , and the Distance EC , draw CF ; the Chord CF is the Diagonal sought, which let from B to G ; so will GA be the Side Sought.

Problem XXVIII.

GIVEN the Area of a rectangled Parallelogram, (36) and the Proportion of the Sides, as 4 to 1 , to find the Sides,

Say $4 : 1 :: 36 : 9$, whose square Root is 3 , the Breadth; as $1 : 4 :: 36 : 144$, whose square Root is 12 , the Length

Problem XXIX.

GIVEN the Difference of the Sides and Sum of the Sides of a right-angled Parallelogram, to find the Sides.

To the half Sum add the half Difference for the Length; from the half Sum take the half Difference for the Breadth

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Problem XXX.

GIVEN the Difference of the Sides, and of their Squares, to find the Sides of a reſtan-gled Parallelogram.

Divide the Difference of the Squares by the Difference of the Sides, the Quotient is the Sum.

Or,

Divide the Difference of the Squares by the Sum of the Sides, the Quotient is the Difference of their Sides, by 6,
2 *Euclid.*

Problem XXXI.

IN any Triangle, the Sum of every two Sides given, to find them ſeverally.

From half the Sum of all the given Numbers, ſubtract each, and the Remainders are the Sides required, *i. e.* each particularly of the Letter wanting.

Problem XXXII

THE Sides of a Trapezium, (A B C D) and one Diagonal (A C) given, to find the other (B D.)

$\square AC + BC - AB$; divide half the Remainder by A C, *Fig. 29.*
the Quotient is C E.

$\square AC + AD - CD$; divide half the Remainder by A C,
the Quotient is A F, by 47. 1. Find B E and D F, add C E and A F, and ſubduct the Sum from A C, the Re-
mainder E F = D G, then $\square DG + \square BG = \square BD$.
Q E. I.

Problem

Problem XXXIII.

IN a right-angled Triangle (ABC), right-angled at B , is given $AC=13$, and the Sum of AB and $BC=17$, to find AB and BC apart.

Fig. 30.

Make the Square $ED=17$ for the Side, and 289 will be the Area; from whence take the Square of $AC=169$, the Remainder $120=2$ Rectangles EB and BD , each of which is a Mean betwixt the Square AB and BC , and double the Triangle ABC .

Wherefore take half $17=8.5$, whose Square is 72.25 ; from which take 60 , and there rest 12.25 , whose Square Root is 3.5 ; which taken from 8.5 , leaves 5 for CB , and added to 8.5 , make 12 for AB .

Like to this is the following. "In a right-angled Triangle ABC , right angled at B , is given $BC=40$, and "the Sum of AB and AC together 150 , to find them "severally."

Square the Perpendicular, and divide that Sum 1600 by double the Sum of AB and $BC=300$, the Quotient is $5\frac{1}{3}$, which added to the half Sum of AB and $BC=75$, gives $80\frac{1}{3}$ for the Hypotenuse AC ; and taken from it, gives $69\frac{2}{3}$ for AB .

Or,

From the Square of the Sum of AB and AC $150=22500$ be subtracted the Square of BC $40=1600$ and divide the Remainder $=20900$ by double the Sum of AB and $BC=300$, the Quotient is $69\frac{2}{3}$ for AB , &c.

Problem XXXIV.

IN the Rectangle ($ABCD$) are given $BE=16$ and $EF=2$. Quare the Area.

Fig. 31.

Seeing the Angles are right, 'tis as $FE:ED::ED:EA$; and again, as $ED:EA::EA:EB$. And seeing E is the third Number, and ED and EA are two Means proportional betwixt 2 and 16 , reduce each into their least Proportion, which is 1 and 8 , and extract the Cube Root of 8 is 2 , which doubled (because 2.16 were halved before) give 4 for ED ; then is EA 8 , and consequently the Area 160 .

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Problem XXXV.

IN the Triangle (A B C) are the three Perpendiculars (A D=56, B F=60, and E C=64 $\frac{1}{3}$) given, to find the three Sides.

In the Triangle, whose Sides are 13, 14, 15, the Per- Fig. 32.
pendicular is found to be 12: wherefore say, as

$$12 : 14 :: \begin{cases} 56 : 65\frac{1}{3} & \text{A B} \\ 60 : 70 & \text{A C} \\ 64\frac{1}{3} : 75\frac{1}{3} & \text{B C.} \end{cases}$$

Problem XXXVI.

IN an Ifofceles Triangle (A B C) are two Circles touching one another; and the Sides of the Triangle, the Diameter of one is 12 and the other 8. Quære the Sides of the Triangle?

E H is the Difference of their Radii=2, and E F the Fig. 33.
Sum of the Radii=10,; then say, if 2 give 10, what will
E G=6 give? Answer, 30 for A E; to which add E D=
6, and it makes 36 for A D, the Perpendicular; then $\square$
A E— $\square$ E G= $\square$ A G; then, as AG : AE :: AD : AC=
A B; then $\square$ A C— $\square$ A D=C D, which doubled, is B C
the Base. Q. E. I.

Problem XXXVII

*THE Hypothenufe A B=48, and the Area of a
right-angled Triangle=384, given to find the
Sides A C and C B.*

On A B=48, the Hypothenufe make a Semi-circle, on Fig. 34.
whose Center D, erect the Perpendicular D G; then double
the Area given 768, which divide by the Hypothenufe=
48, the Quotient is D E=16 of the Ifofceles Triangle
A E B; through E, parallel to A B, draw E C, then A C
and B C are the Sides fought. Q. E. I.

$$\begin{aligned} \square DC - CF \square &= DF \square, \text{ then } A D + D F = A F, \text{ and} \\ \square A F + FC \square &= AC \square, \text{ then } A B - A F = F B, \text{ and} \\ \square C F + FB \square &= B C \square. \end{aligned}$$

Problem

Problem XXXVIII.

THE Diameter of a given Circle = 10, in which make the greatest Rectangle possible, whose Length shall be double its Breadth. Quare the Sides.

- Fig. 35.** Divide the Square of the Diameter by 5, the Quotient is 20, whose Square Root is the Breadth: Deduct 20 out of 100, remains 80, whose Square Root is the Length.
Q E. I.

Problem XXXIX

THE Diameter of a Circle = 8, in it to inscribe on the Diameter an Equilateral Triangle, whose Vertex shall touch the Circumference. Quare its Side.

- Fig. 36.** Cross the Circle with the two Diameters, A B, E F; in it inscribe an Equilateral Triangle A C D, whose Vertex shall be A, so will the Diameter E F cut off the Equilateral A H K. W. W. D.

To find the Side

$$\square AG + \frac{1}{3} \square AG = \square AH.$$

Problem XL.

IN the Circle (A F B G C D) having the Square (A B C D) inscribed, is given each Segment's Area = 224 (A B F) contained betwixt the Side of the Square and the Circumference, to find the Diameter.

- Fig. 37.** Suppose the Diameter = 1, the Area will be $\frac{1}{4}$, whereof $\frac{1}{4}$ will be $\frac{1}{8} = A E B F$, the Area of the Triangle A E B = $\frac{1}{8}$,

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which subtracted from $\frac{11}{8}$ leaves $\frac{1}{4}$ for A B F; then say, as $\frac{1}{4} : \square 1 :: 224$ Quantity given : $\square$ Diameter = 3136, whose square Root is 56 = Diameter A C, supposing the Proportion of the Diameter to the Circumference, as 7 to 22.

Problem XLI.

IN a Circle, whose Diameter (B D) = 65 is a Triangle inscribed (A B C the Side (A B) = 52 and (B C) = 56. Quere the Side (A C.)

$\square B D - \square A B = \square A D$ and $\square B D - \square B C = \square C D$, Fig. 38. then multiply A D by B C, also A B by D C; these two Products = 3900 divide by B D = 65, the Quotient 60 = A C. Q. E. I.

Problem XLII.

IN the Circle (A C E B) is inscribed a Triangle (A B C) A B = 13, B C = 14, A C = 15, and in that Triangle is inscribed a Circle. Quere the two Diameters of both Circles.

Find the Perpendicular A F = 12, then because the Tri- Fig. 39. angles A C E and A B F are similar, say A F : A B :: A C : A E, the Diameter of the circumscribing Circle = $16\frac{1}{4}$. Then seek the Area of the Triangle = 84, which doubled is 168; then add all the three Sides, they make 42, by which divide 168, the Quotient is 4 = G D, half the Diameter of the inscribed Circle.

Then

The Proportion of the Sphere, and of the Five Regular Bodies, inscribed therein, from Peter Herigon, Cursus Math. Vol. I. p. 779.

DIAMETER of the Sphere 2. Circumference of the great Circle 6.28318, Area thereof 3.14159, Area of the Sphere 12.56637, Solidity of it 4.18859.

	<i>Tetraedrum.</i>	<i>Hexaedrum.</i>	<i>Octaedrum.</i>	<i>Dodecahedrum.</i>	<i>Icosihedrum.</i>
Side,	1.6229	1.1547	1.4142	0.7136	1.0514
Surface,	4.6188	8.	6.9282	10.5146	9.5745
Solidity,	0.1513	1.5396	1.3333	2.7851	2.5361

Problem XLIII.

TO find a Cube nearly equal to a given Sphere
(*ABCD.*)

Fig. 40. Quarter the Circle with *AC* and *BD*, and draw *AD* which bisect in *E*; then draw *CE*, the Side of a Cube nearly equal to the Sphere *ABCD.* *W. W. D.*



F I N I S.



